1. (30 points) The following problems are unrelated.

   (a) Evalaute $\int \left[\sqrt{x^5} + \frac{4x^8}{3ax^{2/3}}\right] dx$ where a is a constant.

   (b) Evaluate $\int_0^{\pi/4} \frac{\sec^2 x}{(5 - 2\tan x)^3}\, dx$.

   (c) Find the derivative of $y = 4x^3 + \int_{\sin x}^{x^2} \frac{5t+7}{\sqrt{8-3t}}\, dt$. (Please do not attempt to simplify your final answer.)

Solution:

(a)

$$\int \sqrt{x^5} + \frac{4x^8}{3ax^{2/3}}\, dx = \int x^{5/2} + \frac{4}{3a}x^{22/3}\, dx$$
$$= \frac{2}{7}x^{7/2} + \frac{4}{25a}x^{25/3} + C.$$

(b) We will apply the substitution $u = 5 - 2\tan x$. This yields $-\frac{1}{2}du = \sec^2 x\, dx$ with new limits of integration $u = 5$ and $u = 3$.

$$\int_0^{\pi/4} \frac{\sec^2 x}{(5-2\tan x)^3}\, dx = -\frac{1}{2}\int_5^3 u^{-3}\, du$$
$$= \frac{1}{4u^2}\Big|_5^3$$
$$\frac{4}{225}.$$

(c)

$$\frac{dy}{dx} = 12x^2 + \frac{(5x^2+7)(2x)}{\sqrt{8-3x^2}} - \frac{(5\sin x + 7)\cos x}{\sqrt{8 - 3\sin x}}$$

2. (22 points) Consider the following limit of a Riemann Sum:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{2i}{n}\right)^2 + 1\right]\frac{2}{n}.$$

   (a) Evaluate the limit by first using summation properties and formulas to simplify the sum and then using the methods of chapter 1 to evaluate the limit.

   (b) Determine a definite integral, $\int_a^b f(x)\, dx$, that corresponds to the limit of the provided Riemann sum.

   (c) Find the average value of $f(x)$ over the interval $[a, b]$, where $f(x)$, a, and b, refer to the function and limits of integration you found in (b). (Note that you may use your answer from (a) to help find the solution to this problem.)

Solution:

(a) First, we note that

$$\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^2+1\right]\frac{2}{n} = \sum_{i=1}^{n}\left[\frac{8i^2}{n^3}+\frac{2}{n}\right]$$
$$= \frac{8}{n^3}\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n}\frac{2}{n}$$
$$= \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{2}{n}\cdot n$$
$$= \frac{4(2n^2+3n+1)}{3n^2}+2.$$

So, we have

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^2+1\right]\frac{2}{n} = \lim_{n\to\infty}\left[\frac{4(2n^2+3n+1)}{3n^2}+2\right]$$
$$= \lim_{n\to\infty}\left[\frac{4(2n^2+3n+1)}{3n^2}\cdot\frac{1/n^2}{1/n^2}+2\right]$$
$$= \lim_{n\to\infty}\left[\frac{4(2+3/n+1/n^2)}{3}+2\right]$$
$$= \frac{8}{3}+2$$
$$= \frac{14}{3}.$$

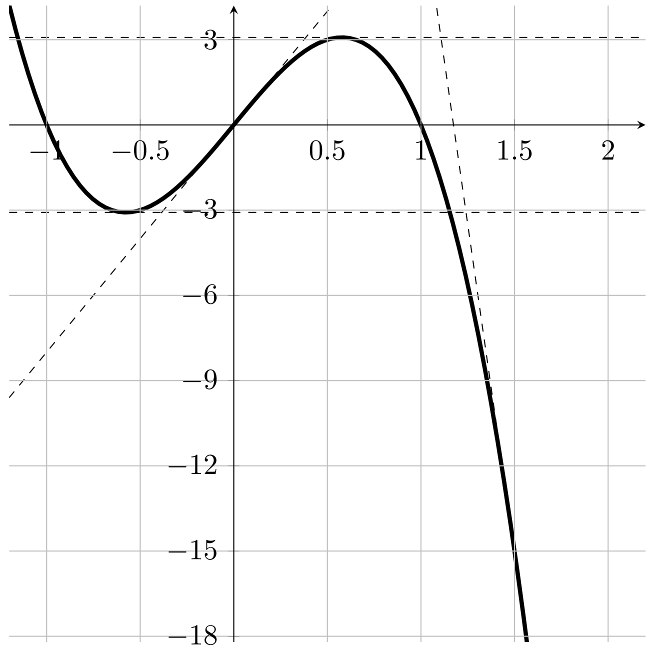
(b) Since $\Delta x = \frac{b-a}{n} = \frac{2}{n}$, we know that $b-a=2$ or $b=a+2$.
Since $a+i\Delta x = \frac{2i}{n}$, we have $a=0$, which means $b=2$. It follows that $f(x)=x^2+1$. So, the integral is

$$\int_0^2 \left[x^2+1\right]\, dx.$$

(c) We have the average value as

$$\frac{1}{2-0}\int_0^2 \left[x^2+1\right]\, dx = \frac{14/3}{2} = \frac{7}{3}.$$

3. (20 points) Consider the graph of $y = g(x)$ provided below. The dashed lines represent the tangent lines of $y = g(x)$ at $x = \pm\frac{1}{\sqrt{3}}, 0$, and $\frac{3}{2}$. Use this graph to answer the questions that follow.



(a) Approximate $\int_{-1}^{3/2} g(x)\,dx$ using the right endpoint rule with five rectangles of equal width.

(b) Is the function $h(x) = \int_0^x g(t)\,dt$ concave up or concave down on $\left(0, \frac{1}{2}\right)$? Justify your answer. If you use a major theorem, state its name.

(c) i. Which of the following initial guesses for Newton's Method will fail to yield an approximation of a solution of $g(x) = 0$? (Circle each that applies. No justification is required.)

$$x = -1/\sqrt{3} \qquad x = 0 \qquad x = 1/\sqrt{3} \qquad x = 1.5$$

ii. Which of the following initial guesses for Newton's Method will most quickly converge to a solution of $g(x) = 0$? (Circle one. No justification is required.)

$$x = -1/\sqrt{3} \qquad x = 0 \qquad x = 1/\sqrt{3} \qquad x = 1.5$$

Solution:

(a)

$$\begin{aligned}\int_{-1}^{3/2} g(x)\,dx &\approx [g(-1/2) + g(0) + g(1/2) + g(1) + g(3/2)]\,\Delta x \\ &= [-3 + 0 + 3 + 0 - 15] \times \frac{1}{2} \\ &= -\frac{15}{2}.\end{aligned}$$

(b) Concave up because $h'(x) = g(x)$ (by FTC Part 1) which implies $h''(x) = g'(x)$, and $g'(x)$ is positive along $(0, 1/2)$ because g is increasing along $(0, 1/2)$.

(c) i. $x = \pm 1/\sqrt{3}$ (The horizontal tangent lines at these points will not intersect the x-axis.)

ii. $x = 0$ (Note that $x = 0$ is itself a solution, so Newton's Method immediately converges to a solution with this initial guess.)

4. (16 pts) Using the grid below, sketch the graph of a **single function**, $y = f(x)$ with each of the following characteristics. (Sketch dashed lines to indicate any asymptotes that are present. The concavity of your graph should be clear.)

f is continuous on its domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$,

$f(-x) = f(x)$ for all x in the domain,

$f(0) = 0$,

$f'(x) > 0$ for $x < -3$, $0 < x < 2$, and $2 < x < 3$,

$f''(x) > 0$ for $-2 < x < 2$,

$f''(x) < 0$ for $x > 2$

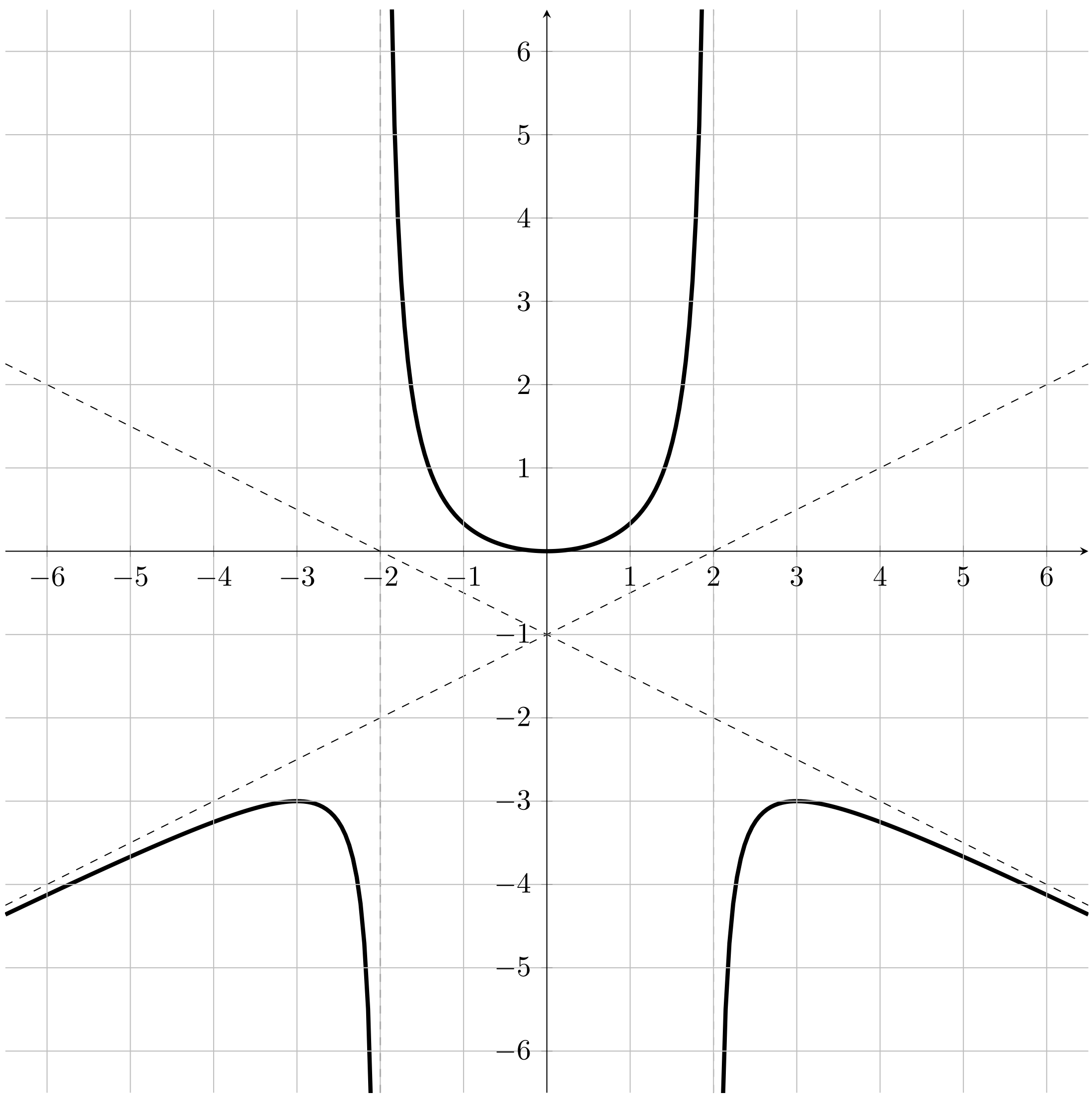
$\lim_{x\to\infty} [f(x) - (-(1/2)x - 1)] = 0$,

$f(3) = -3$

$\lim_{x\to 2^-} f(x) = \infty$,

$\lim_{x\to 2^+} f(x) = -\infty$

Solution:



5. (12 points) A rectangular box with an open top and square base is to have a surface area of 1200 square centimeters. Find the dimensions of such a box with the greatest volume. (Include the correct units in your final answer and be sure to justify that you have found the absolute maximum.)

Solution:

We label the length and width of our box as x, and the height as y. Note that the surface area is

$$1200 = x^2 + 4xy$$

which gives

$$y = \frac{300}{x} - \frac{1}{4}x.$$

The volume is given by

$$V = x^2 y = x^2\left(\frac{300}{x} - \frac{1}{4}x\right) = 300x - \frac{1}{4}x^3.$$

Note that x can be any positive value and this would be a valid box.

We have $V'(x) = 300 - \frac{3}{4}x^2$. When $V'(x) = 0$, we only obtain a solution of $x = 20$ when $x > 0$.

Since $V'(x) > 0$ when $0 < x < 20$ and $V'(x) < 0$ when $x > 20$, then the First Derivative Test for Absolute Extrema tells us that $V(x)$ has an absolute maximum value when $x = 20$. (Alternatively, we can note that $V''(x) = -\frac{3}{2}x < 0$ for all $x > 0$, so the Second Derivative Test for Absolute Extrema implies there is an absolute maximum value when $x = 20$.)

Note that when $x = 20$, we have

$$y = \frac{300}{20} - \frac{1}{4}(20) = 15 - 5 = 10.$$

So, the dimensions of the the box with the greatest volume is

- Width: 20 cm
- Length: 20 cm
- Height: 10 cm