1. (26 pts) Parts (a), (b), and (c) are not related.

   (a) Evaluate $\dfrac{d}{dx}\left[\cos^3(2x+x^{-3})\right]$.

   Solution:

$$\begin{aligned}\frac{d}{dx}\left[\cos^3(2x+x^{-3})\right] &= \frac{d}{dx}\left[\left(\cos(2x+x^{-3})\right)^3\right]\\ &= 3\left(\cos(2x+x^{-3})\right)^2\frac{d}{dx}\left[\cos(2x+x^{-3})\right]\\ &= 3\cos^2(2x+x^{-3})(-\sin(2x+x^{-3}))\frac{d}{dx}\left[2x+x^{-3}\right]\\ &= \boxed{-3\cos^2(2x+x^{-3})\sin(2x+x^{-3})(2-3x^{-4})}\end{aligned}$$

   (b) Evaluate $\dfrac{d}{dx}\left[(x+1)^4(x+2)^5\right]$.

   Express your answer as a fully simplified product of three factors.

   (*Hint:* Identify common factors when simplifying.)

   Solution:

$$\begin{aligned}\frac{d}{dx}\left[(x+1)^4(x+2)^5\right] &= (x+1)^4\left[5(x+2)^4\right]+(x+2)^5\left[4(x+1)^3\right]\\ &= 5(x+1)^4(x+2)^4+4(x+1)^3(x+2)^5\\ &= (x+1)^3(x+2)^4\left[5(x+1)+4(x+2)\right]\\ &= \boxed{(x+1)^3(x+2)^4(9x+13)}\end{aligned}$$

(c) Evaluate $\frac{d}{dx}[\tan x + \csc x]$ by rewriting both $\tan x$ and $\csc x$ in terms of $\sin x$ and $\cos x$ (as applicable), then applying the quotient rule to each quotient in the resulting expression.

Express your final answer in a form that does not include any $\sin x$ or $\cos x$ terms.

Solution:

$$
\begin{aligned}
\frac{d}{dx}[\tan x + \csc x] &= \frac{d}{dx}\left[\frac{\sin x}{\cos x} + \frac{1}{\sin x}\right] \\
&= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} + \frac{\sin x \cdot 0 - 1 \cdot \cos x}{\sin^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} - \frac{\cos x}{\sin^2 x} \\
&= \frac{1}{\cos^2 x} - \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} \\
&= \boxed{\sec^2 x - \csc x \cot x}
\end{aligned}
$$

2. (22 pts) Parts (a) and (b) are not related.

(a) Find the equations of the tangent and normal lines to the curve $x^2 + xy + \sqrt{y} = 7x$ at the point $(1, 4)$.

Solution:

$$\frac{d}{dx}\left[x^2 + xy + y^{1/2}\right] = \frac{d}{dx}[7x]$$

$$2x + (xy' + y) + \frac{1}{2}y^{-1/2}y' = 7$$

$$\left(x + \frac{1}{2\sqrt{y}}\right)y' = 7 - 2x - y$$

$$y' = \frac{7 - 2x - y}{\left(x + \frac{1}{2\sqrt{y}}\right)}$$

The value of y' at $x = 1$ and $y = 4$ is the slope of the tangent line at $(1, 4)$.

$$y'\Big|_{x=1,y=4} = \frac{7 - 2 - 4}{\left(1 + \frac{1}{2\sqrt{4}}\right)} = \frac{1}{1 + 1/4} = \frac{1}{5/4} = \frac{4}{5}$$

Equation of tangent line: $\boxed{y - 4 = \frac{4}{5}(x - 1)}$

Equation of normal line: $\boxed{y - 4 = -\frac{5}{4}(x - 1)}$

(b) Find all values of x at which the curve $y = 3x^{7/3} - 6x^{4/3} + 1$ has a horizontal tangent line.

Solution:

$$y'(x) = \frac{d}{dx}\left[3x^{7/3} - 6x^{4/3} + 1\right] = 3 \cdot \frac{7}{3}x^{4/3} - 6 \cdot \frac{4}{3}x^{1/3} + 0$$

$$= 7x^{4/3} - 8x^{1/3}$$

$$= x^{1/3}(7x - 8)$$

The curve has a horizontal tangent line at every value of x for which $y'(x) = 0$.

$$x^{1/3}(7x - 8) = 0$$

$$x = \boxed{0, 8/7}$$

3. (26 pts) Parts (a) and (b) are not related.

(a) The position function of a particle P is given by $s(t) = \cos t - \cos^2 t$, where s is in meters, t is in seconds, and $0 \leq t \leq \pi$.

i. Identify particle P's acceleration function, $a(t)$, including the correct unit of measurement. You do **not** need to algebraically simplify your final answer.

Solution:

$$v(t) = s'(t) = -\sin t - 2\cos t(-\sin t) = -\sin t + 2\sin t\cos t$$

$$a(t) = v'(t) = -\cos t + 2\left[\sin t \cdot (-\sin t) + \cos t \cdot \cos t\right]$$

$$= \boxed{-\cos t + 2(\cos^2 t - \sin^2 t)\ \text{m/s}^2}$$

ii. Find the distance traveled by particle P between $t = 0$ and $t = \pi$ seconds, including the correct unit of measurement. Fully simplify your final answer.

Solution:

First, we need to identify the values of t on the interval $(0, \pi)$ at which the particle is at rest.

$$v(t) = -\sin t + 2\sin t\cos t = \sin t\,(2\cos t - 1) = 0$$

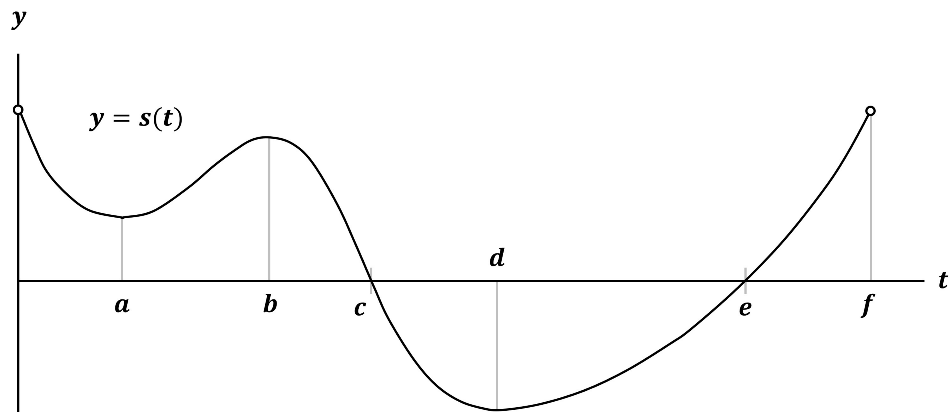
The equation $\sin t = 0$ has no solutions on the interval $(0, \pi)$.

$$\begin{aligned} 2\cos t - 1 &= 0 \\ \cos t &= 1/2 \\ t &= \pi/3 \end{aligned}$$

The sign of $v(t)$ changes at $t = \pi/3$, so that the total distance traveled by the particle between $t = 0$ and $t = \pi$ seconds is

$$\begin{aligned} D &= |s(\pi/3) - s(0)| + |s(\pi) - s(\pi/3)| \\ &= |(\cos(\pi/3) - \cos^2(\pi/3)) - (\cos(0) - \cos^2(0))| + |(\cos(\pi) - \cos^2(\pi)) - (\cos(\pi/3) - \cos^2(\pi/3))| \\ &= |(1/2 - 1/4) - (1 - 1)| + |(-1 - 1) - (1/2 - 1/4)| \\ &= |1/4 - 0| + |-2 - 1/4| \\ &= 1/4 + 9/4 = 10/4 = \boxed{5/2\ \text{m}} \end{aligned}$$

(b) A graph of the position function for a particle Q for time t on the open interval $(0, f)$ is provided below.



Answer the following questions, using t values 0, a, b, c, d, e, and f, as appropriate.

i. On what time interval(s), if any, is the velocity of particle Q positive? Express your answer using interval notation. You do not need to provide any justification for your answer.

Solution:

$\boxed{(a, b) \cup (d, f)}$

($v(t) = s'(t)$, and the graph indicates that $s(t)$ is increasing on those intervals.)

ii. At what value(s) of t, if any, is particle Q at rest? You do not need to provide any justification for your answer.

Solution:

$\boxed{t = a, b, d}$

(The graph indicates that $s(t)$ has a horizontal tangent line at those three t values.)

iii. On time interval (b, d), does particle Q's **speed** increase, decrease, or both? Explain your answer.

Solution:

$\boxed{\text{Both}}$

It was determined in part (ii) that the particle is at rest at $t = b$ and at $t = d$. The graph indicates that the particle's position at $t = b$ and at $t = d$ are different, which means the particle moved on time interval (b, d). Therefore, since the particle starts at rest and ends at rest for interval (b, d), its speed must have increased then decreased.

4. (26 pts) Parts (a) and (b) are not related.

(a) Consider the function $f(x)$, defined as follows:

$$f(x) = \begin{cases} x^2 & , \quad x < 1 \\ 2x^2 - 2x + 1 & , \quad x \geq 1 \end{cases}$$

i. Is $f(x)$ continuous at $x = 1$? Fully justify your answer using the limit definition of continuity.

Solution:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x^2 - 2x + 1) = 2 - 2 + 1 = 1$$

$$f(1) = 2 \cdot 1^2 - 2 \cdot 1 + 1 = 2 - 2 + 1 = 1$$

Since $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) = f(1)$, $\boxed{f(x) \text{ is continuous at } x = 1}$

ii. Is $f(x)$ differentiable at $x = 1$? Fully justify your answer using the limit definition of $f'(1)$.

Solution:

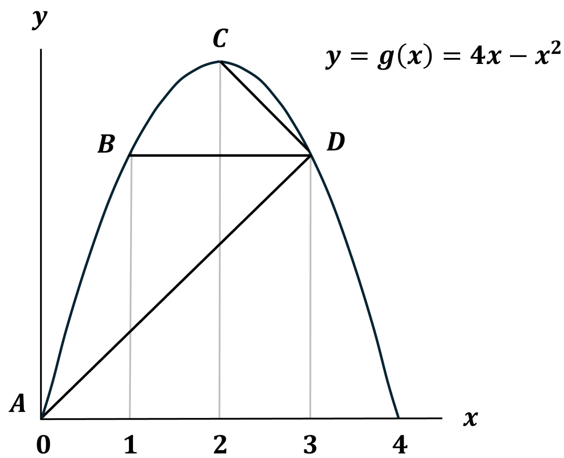
$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1^-} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1^-} (x + 1) = 2$$

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{(2x^2 - 2x + 1) - 1}{x - 1} = \lim_{x \to 1^+} \frac{2x(x-1)}{x - 1} = \lim_{x \to 1^+} 2x = 2$$

Since $\lim\limits_{x \to 1^-} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1^+} \dfrac{f(x) - f(1)}{x - 1}$, then $f'(1)$ exists.

Therefore, $\boxed{f(x) \text{ is differentiable at } x = 1}$

(b) The graph below depicts the curve $y = g(x) = 4x - x^2$. Let A, B, C, and D be points on the curve at $x = 0$, $x = 1$, $x = 2$, and $x = 3$, respectively.



i. Find the slope of the secant lines AD, BD, and CD. Show the details of your calculations and clearly indicate which secant line corresponds to each slope value.

Solution:

The coordinates of points A, B, C, and D are as follows:

$$g(0) = 0 - 0 = 0 \quad \Rightarrow \quad \text{coordinates of point A: } (0,0)$$
$$g(1) = 4 - 1 = 3 \quad \Rightarrow \quad \text{coordinates of point B: } (1,3)$$
$$g(2) = 8 - 4 = 4 \quad \Rightarrow \quad \text{coordinates of point C: } (2,4)$$
$$g(3) = 12 - 9 = 3 \quad \Rightarrow \quad \text{coordinates of point D: } (3,3)$$

The slopes of the secant lines are as follows:

$$\text{slope of secant line AD} = \frac{3-0}{3-0} = \boxed{1}$$
$$\text{slope of secant line BD} = \frac{3-3}{3-1} = \boxed{0}$$
$$\text{slope of secant line CD} = \frac{3-4}{3-2} = \boxed{-1}$$

ii. Use the limit definition of the derivative to determine the slope of the line that is tangent to the given curve at point D. You must evaluate an appropriate limit to earn full credit.

Solution:

$$\begin{aligned}
\lim_{x\to 3} \frac{f(x)-f(3)}{x-3} &= \lim_{x\to 3} \frac{(4x-x^2)-3}{x-3} \\
&= \lim_{x\to 3} \frac{-x^2+4x-3}{x-3} \\
&= \lim_{x\to 3} \frac{-(x^2-4x+3)}{x-3} \\
&= \lim_{x\to 3} \frac{-(x-1)(x-3)}{x-3} \\
&= \lim_{x\to 3} -(x-1) = -(3-1) = \boxed{-2}
\end{aligned}$$