Let u and w denote positive real numbers, then:

(a) $\log_b(u) = \dfrac{\log_a(u)}{\log_a(b)}$ for $a > 0$, $a \neq 1$.

(b) $A = \dfrac{1}{2}r^2\theta$

(c) $S = r\theta$

1. Determine the end behavior for the rational function: $R(x) = \dfrac{2x^4 + 3x^3 - 2x^2}{x^3 - 2x - 1}$. (5 pts)

Solution:

$$\begin{array}{r|l} & 2x + 3 \\ x^3 - 2x - 1 & 2x^4 + 3x^3 - 2x^2 + 0x + 0 \\ & -(2x^4 - 4x^2 - 2x) \\ & 3x^3 + 2x^2 + 2x + 0 \\ & -(3x^3 - 6x - 3) \\ & 2x^2 + 8x + 3 \end{array}$$

So the slant asymptote is $\boxed{y = 2x + 3}$.

In arrow notation:

$$R(x) \to 2x + 3 \text{ as } x \to -\infty \quad (1)$$
$$R(x) \to 2x + 3 \text{ as } x \to \infty \quad (2)$$

2. Consider the following rational function: $r(x) = \dfrac{x^3 - 4x}{x^3 - 6x^2 + 8x}$. Answer the following: (12 pts)

(a) Find the x-coordinate of any hole(s) of $r(x)$. If there are none write NONE.

Solution:

We will need the factored form of the function to proceed:

$$r(x) = \frac{x(x^2 - 4)}{x(x^2 - 6x + 8)} \quad (3)$$
$$= \frac{x(x + 2)(x - 2)}{x(x - 4)(x - 2)} \quad (4)$$

Our function reduces to the simplified form:

$$y = \frac{x + 2}{x - 4} \quad (5)$$

As there are cancellations of the $x - 2$ terms and x terms, there are holes at:

$$\boxed{x = 2} \quad (6)$$
$$\boxed{x = 0} \quad (7)$$

(b) Find the y-coordinate of any hole(s) you found in part (a). If there are none write NONE.

Solution:

To find the y coordinate of the hole at $x = 2$, we substitute the location of our hole, $x = 2$, into the simplified version of $r(x)$:

$$y(2) = \frac{(4)}{(-2)} \tag{8}$$
$$= \boxed{-2} \tag{9}$$

To find the y coordinate of the hole at $x = 0$, we substitute the location of our hole, $x = 0$, into the simplified version of $r(x)$:

$$y(0) = \frac{(2)}{(-4)} \tag{10}$$
$$= \boxed{-\frac{1}{2}} \tag{11}$$

So the locations of our two holes are $(2, -2)$ and $(0, -\frac{1}{2})$.

(c) Find all vertical asymptote(s) of $r(x)$. If there are none write NONE.

Solution:

There is a vertical asymptote where the simplified rational function's denominator goes to 0, so we have:

$$\text{Vertical asymptote: } \boxed{x = 4} \tag{12}$$

(d) Determine the end behavior of $r(x)$ and fill in the blanks: $r(x) \to$ ____ as $x \to -\infty$ and $r(x) \to$ ____ as $x \to \infty$.

Solution:

Taking the ratio of the leading terms we get $r(x) \approx \dfrac{x^3}{x^3} = 1$ as $x \to \pm\infty$. So we get:

$$\text{Horizontal asymptote: } \boxed{y = 1} \tag{13}$$

In arrow notation, the end behavior can be written as:

$$r(x) \to \boxed{1} \text{ as } x \to -\infty \tag{14}$$
$$r(x) \to \boxed{1} \text{ as } x \to \infty \tag{15}$$

(e) Find all x-intercept(s) of $r(x)$. If there are none write NONE.

Solution:

The only x-intercept is found when $r(x) = 0$. Using the simplified function:

$$\frac{x+2}{x-4} = 0 \tag{16}$$
$$x + 2 = 0 \tag{17}$$
$$x = -2 \tag{18}$$

So the x-intercept is $\boxed{x = -2}$.

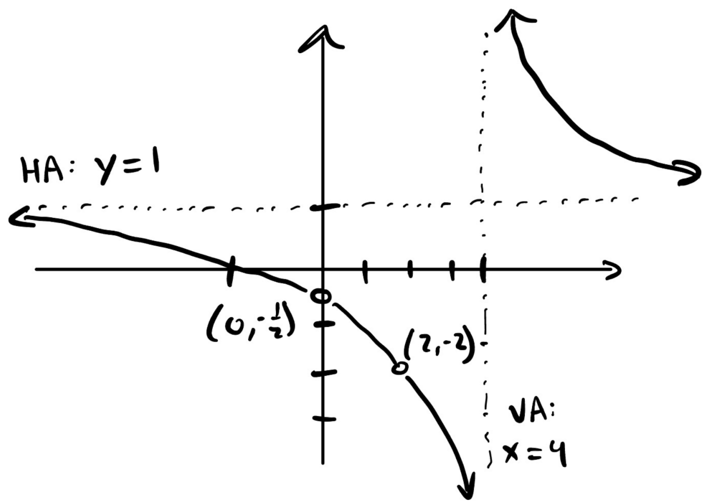
(f) Find the y-intercept. If there is none write NONE.

Solution:

Since there is a hole at $x = 0$ then there is no y-intercept: $\boxed{\text{NONE}}$.

(g) Sketch the graph of $r(x)$ using parts (a)-(f). **Label** all intercept(s), hole(s), and asymptote(s) as relevant.

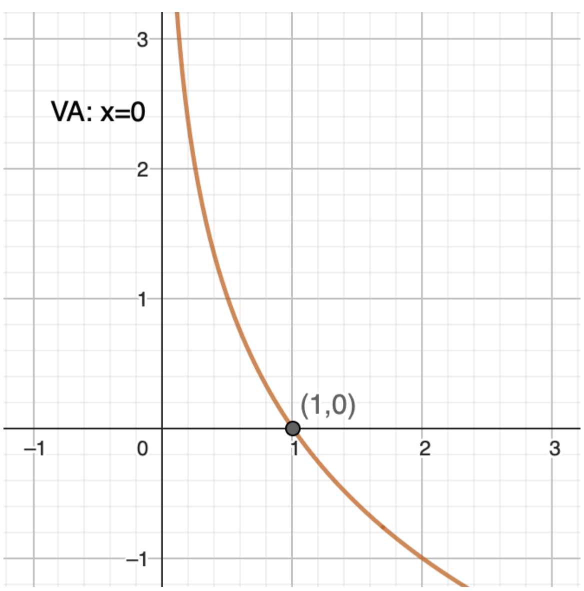
Solution:



3. Sketch the following graphs: Be sure to **label** any asymptotes for each graph. (10 pts)

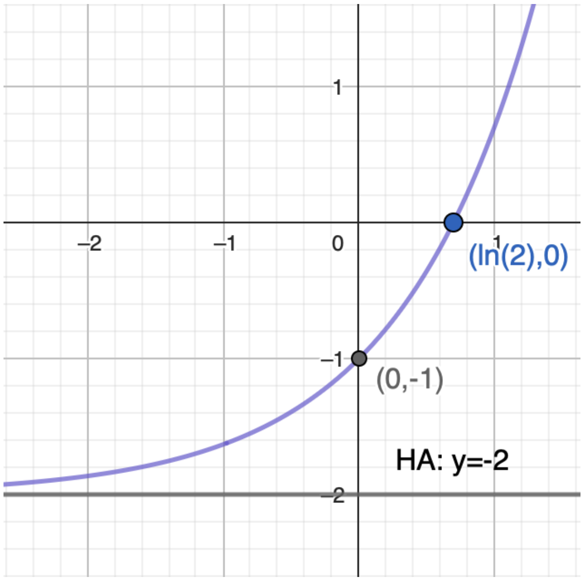
(a) $f(x) = -\log_2(x)$

Solution: This function is $\log_2(x)$ flipped about the x-axis:



(b) $g(x) = e^x - 2$

Solution: This function is e^x shifted by -2 vertically:



(c) Use the function given in part (b) of this question to determine the value: $g\left(\ln\left(\frac{1}{2}\right)\right)$.

Solution:

$$g\left(\ln\left(\frac{1}{2}\right)\right) = e^{\ln\left(\frac{1}{2}\right)} - 2 \tag{19}$$

$$= \frac{1}{2} - 2 \tag{20}$$

$$= \boxed{-\frac{3}{2}} \tag{21}$$

(d) For $g(x) = e^x - 2$ graphed in part (b) fill in the blanks for the following:

$g(x) \to$ ____ as $x \to -\infty$ and $g(x) \to$ ____ as $x \to \infty$.

Solution:

$g(x) \to \boxed{-2}$ as $x \to -\infty$ and $g(x) \to \boxed{\infty}$ as $x \to \infty$.

4. The following are unrelated.

(a) Simplify (rewrite without logs): $\ln(e) - \log_3(27) + 8^{2\log_8 5} + \log(1)$ (4 pts)

Solution:

$$\ln(e) - \log_3(27) + 8^{2\log_8 5} + \log(1) = 1 - \log_3(3^3) + 8^{\log_8 5^2} + 0 \tag{22}$$

$$= 1 - 3 + 5^2 \tag{23}$$

$$= \boxed{23} \tag{24}$$

(b) Rewrite as a single logarithm: $5\log_2(x) - \frac{1}{2}(\log_2(y) + 7\log_2(x))$ (4 pts)

Solution:

$$\begin{aligned}
5\log_2(x) - \frac{1}{2}(\log_2(y) + 7\log_2(z)) &= \log_2\left(x^5\right) - \frac{1}{2}\left(\log_2(y) + \log_2\left(x^7\right)\right) && (25)\\
&= \log_2\left(x^5\right) - \frac{1}{2}\log_2\left(yx^7\right) && (26)\\
&= \log_2\left(x^5\right) - \log_2\left(y^{1/2}x^{7/2}\right) && (27)\\
&= \log_2\left(\frac{x^5}{y^{1/2}x^{7/2}}\right) && (28)\\
&= \boxed{\log_2\left(\frac{x^{3/2}}{y^{1/2}}\right)} && (29)
\end{aligned}$$

5. The half-life of Polonium-210 is 140 days. Suppose a sample of this substance has a mass of 130 mg. Use this information to help answer the following: (10 pts)

(a) How many days until 65 mg remains?

Solution:

After 140 days, half of the original mass remains. Hence $\frac{1}{2}(130) = 65$ mg remains after $\boxed{140 \text{ days}}$

(b) Find a function $m(t) = m_\circ 2^{-t/h}$ that models the mass remaining after t days.

Solution:

Here $m_0 = 130$ and $h = 140$. Hence

$$\boxed{m(t) = (130)2^{-t/140}}$$

(c) Find a function $m(t) = m_\circ e^{rt}$ that models the mass remaining after t days.

Solution:

One method for finding r is equating our model from part (b) with our model from part (c) and solving for r.

$$\begin{aligned}
m_\circ e^{rt} &= m_\circ 2^{-t/h} && (30)\\
e^{rt} &= 2^{-t/h} && (31)\\
rt &= \ln\left(2^{-\frac{t}{h}}\right) && (32)\\
rt &= -\frac{t}{h}\ln 2 && (33)\\
r &= -\frac{\ln 2}{h} && (34)
\end{aligned}$$

Hence a model of the remaining mass would be $\boxed{m(t) = 130e^{-\frac{\ln 2}{140}t}}$ since $h = 140$ and $m_\circ = 130$.

(d) According to your model, how much of the mass remains after 30 days? Give the exact answer, do not attempt to round (approximate) your answer.

Solution:

The amount of mass remaining after 30 days would be:

$(130)e^{-\frac{\ln 2}{140}30} = \boxed{(130)e^{-\frac{3\ln 2}{14}} \text{ mg}}$

6. Solve the following equations for x. Do not attempt to round (approximate) your answers: (16 pts)

(a) $e^{0.7} = e^{x^2-1}$

Solution:

Using the one-to-one property of exponential functions

$$e^{0.7} = e^{x^2-1} \quad (35)$$
$$0.7 = x^2 - 1 \quad (36)$$
$$x^2 = 1.7 \quad (37)$$
$$x = \boxed{\pm\sqrt{1.7}} \quad (38)$$

(b) $\log 6 = \log x + \log(x-1)$

Solution:

$$\log 6 = \log x + \log(x-1) \quad (39)$$
$$\log 6 = \log x(x-1) \quad (40)$$
$$6 = x(x-1) \quad (41)$$
$$x^x - x - 6 = 0 \quad (42)$$
$$(x-3)(x+2) = 0 \quad (43)$$
$$x = -2, 3 \quad (44)$$

However, $x = -2$ is not in the domain of $\log_3(x)$ and hence is an extraneous solution. Also, plugging in $x = 3$ to both sides of the equation, we find that both sides match. Hence the valid solution is

$$\boxed{x = 3}$$

(c) $5^{2x} = 3^{7x+2}$

Solution:

We take $\log_3$ of both sides of the equation (other base logs will work too):

$$5^{2x} = 3^{7x+2} \quad (45)$$
$$\log_3(5^{2x}) = \log_3(3^{7x+2}) \quad (46)$$
$$2x \log_3 5 = 7x + 2 \quad (47)$$
$$2x \log_3 5 - 7x = 2 \quad (48)$$
$$x(2 \log_3 5 - 7) = 2 \quad (49)$$
$$x = \boxed{\frac{2}{2 \log_3 5 - 7}} \quad (50)$$

(d) $\log_2(x-4) = 3 + \log_2 5$

Solution:

$$\log_2(x-4) = 3 + \log_2 5 \quad (51)$$
$$\log_2(x-4) - \log_2 5 = 3 \quad (52)$$
$$\log_2\left(\frac{x-4}{5}\right) = 3 \quad (53)$$
$$\frac{x-4}{5} = 2^3 \quad (54)$$
$$x - 4 = 40 \quad (55)$$
$$x = \boxed{44} \quad (56)$$

7. The area of a sector of a circle with a central angle of $\frac{7\pi}{5}$ radians is 6 m^2. Find the radius of the circle. (Give your answer in exact form, do not attempt to round (approximate).) (4 pts)

Solution:

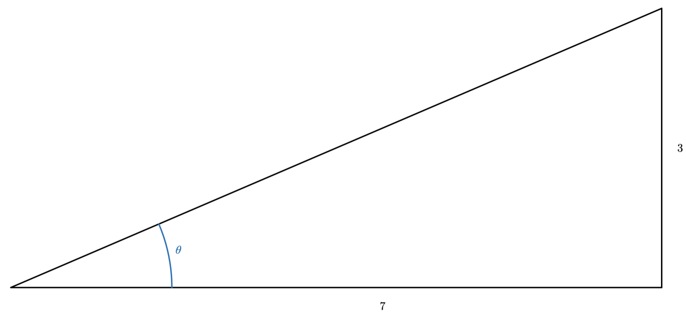
We know that the area of a sector of a circle or radius r with a central angle θ is given by $A = \frac{1}{2}r^2\theta$. Hence we can write

$$\frac{1}{2}r^2\left(\frac{7\pi}{5}\right) = 6 \quad (57)$$
$$r^2\left(\frac{7\pi}{5}\right) = 12 \quad (58)$$
$$r^2 = \frac{60}{7\pi} \quad (59)$$
$$r = \boxed{\sqrt{\frac{60}{7\pi}} \text{ m}} \quad (60)$$

8. Suppose $\tan\theta = \frac{3}{7}$. Use this information to help answer the following: (8 pts)

(a) Sketch a triangle that has acute angle θ.

Solution:



(b) Find $\sin\theta$.

The length of the hypotenuse, h, can be found by using the Pythagorean theorem: $h^2 = 7^2 + 3^2$ so $h = \sqrt{58}$.

Thus $\boxed{\sin\theta = \frac{3}{\sqrt{58}}}$

(c) Find $\cot\theta$.

$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{3}{7}} = \boxed{\frac{7}{3}}$

9. If you know $\sin\theta < 0$ and $\cos\theta = \frac{1}{3}$ what quadrant must θ be in when graphed in standard position? No justification is required for this problem. (3 pts)

Solution:

$\sin\theta < 0$ in quadrants III and IV and $\cos\theta > 0$ in quadrants I and IV so θ must be in $\boxed{\text{Quadrant IV.}}$

10. Find the exact value of the trigonometric function:

(a) $\sin(\pi)$ (3 pts)

(b) $\cos\left(\frac{5\pi}{3}\right)$ (3 pts)

Solution:

$\sin(\pi) = \boxed{0}$

$\cos\left(\frac{5\pi}{3}\right) = \boxed{\frac{1}{2}}$

(c) $\sin\left(-\frac{3\pi}{4}\right)$ (3 pts)

Solution:

$$\sin\left(-\frac{3\pi}{4}\right) = \boxed{-\frac{\sqrt{2}}{2}}$$

(d) $\csc(120^o)$ (3 pts)

$$\csc(120^o) = \frac{1}{\sin(120^o)} = \frac{1}{-\frac{\sqrt{3}}{2}} = \boxed{\frac{2}{\sqrt{3}}}$$

(e) $\tan\left(\frac{11\pi}{6}\right)$ (3 pts)

Solution:

$$\tan\left(\frac{11\pi}{6}\right) = \frac{\sin\left(\frac{11\pi}{6}\right)}{\cos\left(\frac{11\pi}{6}\right)} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \boxed{-\frac{1}{\sqrt{3}}}$$

11. Evaluate (express your answer as a single fraction): (4 pts)

$$2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)$$

Solution:

$$2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) = 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} \quad (61)$$

$$= \frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{4} \quad (62)$$

$$= \boxed{\frac{2\sqrt{6}-\sqrt{2}}{4}} \quad (63)$$

12. A 11 foot ladder leans against the vertical wall of a shed so that the upper end of the ladder just reaches the roof. If the ladder forms a 30° angle with the ground, how high is the roof? (5 pts)

Solution:

$$\sin(30^\circ) = \frac{h}{11} \quad (64)$$

$$h = 11\sin(30^\circ) \quad (65)$$

$$= 11\cdot\frac{1}{2} \quad (66)$$

$$= \boxed{\frac{11}{2}\text{ ft}} \quad (67)$$