

1. The following four problems are not related. (*Reminder:* You may not use L'Hopital's Rule or "dominance of powers" arguments for any problem on this exam.)

- (a) (10 pt) Find the value of $\lim_{t \rightarrow \infty} \frac{t^{-1} - t^{3/2} + 2}{t^{-1} + 5\sqrt{t^3} + 4}$ or explain why the limit does not exist.

Solution: Divide each term by $\sqrt{t^3}$, the highest power in the denominator.

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{t^{-1} - \sqrt{t^3} + 2}{t^{-1} + 5\sqrt{t^3} + 4} &= \lim_{t \rightarrow \infty} \frac{\frac{1}{t} - \sqrt{t^3} + 2}{\frac{1}{t} + 5\sqrt{t^3} + 4} \cdot \frac{\frac{1}{\sqrt{t^3}}}{\frac{1}{\sqrt{t^3}}} \\ &= \lim_{t \rightarrow \infty} \frac{\frac{1}{t^{5/2}} - 1 + \frac{2}{\sqrt{t^3}}}{\frac{1}{t^{5/2}} + 5 + \frac{4}{\sqrt{t^3}}} \\ &= \frac{0 - 1 + 0}{0 + 5 + 0} = \boxed{-\frac{1}{5}} \end{aligned}$$

- (b) (10 pt) Let $h(x) = \frac{7 \sin x}{9x + \sin x}$. Find $h'(\pi)$. Fully simplify your answer.

Solution:

The derivative of the function can be found using the quotient rule.

$$h'(x) = \frac{(9x + \sin x)(7 \cos x) - (7 \sin x)(9 + \cos x)}{(9x + \sin x)^2}.$$

The value of the derivative at $x = \pi$ is

$$\begin{aligned} h'(\pi) &= \frac{(9\pi + 0) \cdot 7(-1) - 0}{(9\pi + 0)^2} \\ &= \frac{(9\pi)(-7)}{(9\pi)^2} = \boxed{-\frac{7}{9\pi}}. \end{aligned}$$

- (c) (10 pt) Find dy/dx given $\frac{3}{4}(x + y)^{4/3} = 1 + x^2y$.

Solution: Use implicit differentiation. Apply the chain rule to the left side of the equation and the product rule to the right side.

$$\begin{aligned} \frac{3}{4}(x + y)^{4/3} &= 1 + x^2y \\ (x + y)^{1/3} \left(1 + \frac{dy}{dx}\right) &= 0 + x^2 \frac{dy}{dx} + 2xy \end{aligned}$$

Now expand the left side, gather dy/dx terms, then solve for dy/dx .

$$\begin{aligned} \sqrt[3]{x + y} + \sqrt[3]{x + y} \frac{dy}{dx} &= x^2 \frac{dy}{dx} + 2xy \\ \sqrt[3]{x + y} \frac{dy}{dx} - x^2 \frac{dy}{dx} &= 2xy - \sqrt[3]{x + y} \end{aligned}$$

$$\frac{dy}{dx} (\sqrt[3]{x+y} - x^2) = 2xy - \sqrt[3]{x+y}$$

$$\frac{dy}{dx} = \frac{2xy - \sqrt[3]{x+y}}{\sqrt[3]{x+y} - x^2} = \frac{\sqrt[3]{x+y} - 2xy}{x^2 - \sqrt[3]{x+y}}$$

(d) (10 pt) Let $f(x) = |x| - x^3$.

- Write the limit definition of $f'(a)$, the derivative of $f(x)$ at $x = a$.
- Use the limit definition to find the value of $f'(0)$ if it exists, or show why it does not exist.

Solution:

This problem is similar to Written HW 5 #5b.

- $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ OR $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$
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$$\begin{aligned} f'(0) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{|h| - h^3 - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{|h| - h^3}{h} \end{aligned}$$

Approaching 0 from the right, substitute $|h| = h$. Then the limit value is

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{|h| - h^3}{h} &= \lim_{h \rightarrow 0^+} \frac{h - h^3}{h} \\ &= \lim_{h \rightarrow 0^+} (1 - h^2) = 1. \end{aligned}$$

Approaching 0 from the left, substitute $|h| = -h$. Then the limit value is

$$\begin{aligned} \lim_{h \rightarrow 0^-} \frac{|h| - h^3}{h} &= \lim_{h \rightarrow 0^-} \frac{-h - h^3}{h} \\ &= \lim_{h \rightarrow 0^-} (-1 - h^2) = -1. \end{aligned}$$

Because the one-sided limit values are not equal, $f'(0)$ does not exist.

Alternate Solution: Using the alternate formula for $f'(a)$, we have

$$\begin{aligned} f'(0) &= \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{|x| - x^3 - 0}{x} \\ &= \lim_{x \rightarrow 0} \frac{|x| - x^3}{x} \end{aligned}$$

The solution then proceeds as described above.

2. (18 pt) Let $g(x) = \frac{x^2 + 16}{3 - x}$.

- (a) Does the graph of $g(x)$ have any vertical asymptotes? If so, find the asymptotes and justify your answer using limits. If not, explain why not.

Solution: Check $x = 3$, the only value that produces a zero denominator. Approaching from the right,

$$\lim_{x \rightarrow 3^+} \frac{x^2 + 16}{3 - x} = -\infty$$

because the numerator approaches a positive number and the denominator approaches zero with negative values. Therefore there is a vertical asymptote at $x = 3$.

Alternatively, approaching 3 from the left,

$$\lim_{x \rightarrow 3^-} \frac{x^2 + 16}{3 - x} = \infty.$$

- (b) Find all critical numbers of $g(x)$, if any.

Solution: First differentiate g using the quotient rule.

$$\begin{aligned} g(x) &= \frac{x^2 + 16}{3 - x} \\ g'(x) &= \frac{(3 - x)(2x) - (x^2 + 16)(-1)}{(3 - x)^2} \\ &= \frac{-x^2 + 6x + 16}{(3 - x)^2} \end{aligned}$$

Next solve $g'(x) = 0$.

$$\begin{aligned} \frac{-x^2 + 6x + 16}{(3 - x)^2} &= 0 \\ -x^2 + 6x + 16 &= 0 \\ x^2 - 6x - 16 &= 0 \\ (x + 2)(x - 8) &= 0 \\ x &= -2, 8 \end{aligned}$$

The critical numbers of $g(x)$ are -2 and 8 . Note that although $g'(3)$ does not exist, 3 is not a critical number because it is not in the domain of the function.

3. (10 pt) The line $y = -6x + 17$ is tangent to $y = bx^3 + c$ at $(2, 5)$. Find the values of the constants b and c .

Solution:

This problem is similar to Written HW 5 #9.

Substituting the point $(2, 5)$ into the cubic function gives

$$y(2) = b(2^3) + c = 5 \implies 8b + c = 5.$$

The derivative of the function is

$$y' = 3bx^2.$$

The slope of the tangent line at $x = 2$ is -6 , so

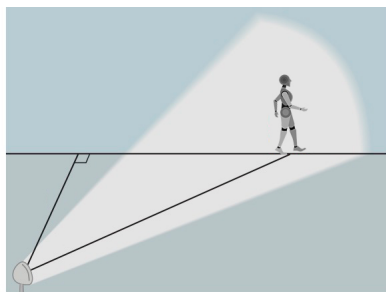
$$y'(2) = -6 \implies 3b(2^2) = 12b = -6 \implies \boxed{b = -\frac{1}{2}}.$$

Then solve for c .

$$8b + c = 5 \implies 8\left(-\frac{1}{2}\right) + c = 5 \implies \boxed{c = 9}.$$

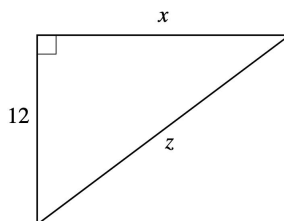
The cubic function is $y = -\frac{1}{2}x^3 + 9$.

4. (14 pt) A light is located on the ground 12 ft from a straight path. A robot begins walking along the path at a speed of 5 ft/sec, starting from the point on the path closest to the light. At what rate is the distance between the robot and the light changing when the robot is 16 ft from its starting point?



Solution:

This problem is similar to WebAssign HW 2.7 #2.



Let x be the distance between the robot and its starting point. Let z be the distance between the robot and the light. It is given that $dx/dt = 5$ ft/sec and we are asked to find dz/dt when $x = 16$ ft. By the Pythagorean Theorem,

$$x^2 + 12^2 = z^2.$$

Differentiate both sides with respect to t to obtain

$$\begin{aligned} 2x \frac{dx}{dt} + 0 &= 2z \frac{dz}{dt} \\ x \frac{dx}{dt} &= z \frac{dz}{dt}. \end{aligned}$$

Next substitute $x = 16$, $z = \sqrt{12^2 + 16^2} = \sqrt{400} = 20$, and $dx/dt = 5$.

$$16 \cdot 5 = 20 \frac{dz}{dt}$$

Then solve for dz/dt .

$$\frac{dz}{dt} = \frac{16 \cdot 5}{20} = \boxed{4 \text{ ft/sec}}$$

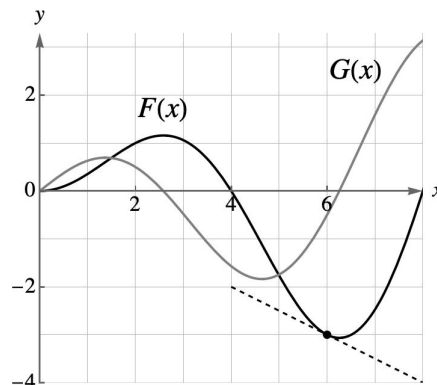
5. (18 pt) The graphs of functions $F(x)$ and $G(x)$ are shown below, along with the line tangent to $y = F(x)$ at $x = 6$. The tangent line passes through three grid intersection points.

- (a) Find the linear approximation of $F(x)$ centered at $a = 6$.
Use it to approximate the value of $F(5.96)$.

- (b) The line tangent to $y = G(x)$ at $x = 4$ has the equation $\pi x + 4y = 2\pi$. Find

$$\lim_{h \rightarrow 0} \frac{G(4+h) + \pi/2}{h}$$

or explain why the limit does not exist.



Solution:

- (a) The linear approximation of $F(x)$ at $x = 6$ is

$$L(x) = F(6) + F'(6)(x - 6).$$

The graph shows that $F(6) = -3$ and $F'(6) = -\frac{1}{2}$, the slope of the tangent line. Therefore

$$L(x) = -3 - \frac{1}{2}(x - 6).$$

An approximation for $F(5.96)$ is

$$\begin{aligned} L(5.96) &= -3 - \frac{1}{2}(5.96 - 6) \\ &= -3 - \frac{1}{2}(-0.04) = \boxed{-2.98}. \end{aligned}$$

- (b) *This problem is related to WebAssign HW 2.1 #4.*

The equation of the tangent line can be written as $y = -\frac{\pi}{4}x + \frac{\pi}{2}$, so $G(4) = -\frac{\pi}{4}(4) + \frac{\pi}{2} = -\frac{\pi}{2}$. The expression

$$\lim_{h \rightarrow 0} \frac{G(4+h) + \pi/2}{h} = \lim_{h \rightarrow 0} \frac{G(4+h) - G(4)}{h}$$

is the definition of $G'(4)$, which equals $\boxed{-\pi/4}$, the slope of the tangent line.

- (c) One of the two functions shown is the derivative of the other. No justification is necessary for parts (i)–(ii).

i. Which function (F or G) is the derivative? _____

ii. For each statement below, write the word True or False.

- _____ $F'(a) = G(a)$ for every a in the domain of both functions.
- _____ $G'(a) = F(a)$ for every a in the domain of both functions.
- _____ $dF/dx > 0$ on every interval where $G > 0$.
- _____ $dG/dx < 0$ on every interval where $F < 0$.

Solution:

(c) *This problem is related to Written HW 5 #4.*

The intervals where F has positive slope, negative slope, and zero slope correspond to the intervals where G is positive, negative, and zero, respectively, so G is the derivative of F . It follows that the first and third statements below are true, as confirmed by the graph.

- ☐ True: $F'(a) = G(a)$ for every a in the domain of both functions.
- ☐ False: $G'(a) = F(a)$ for every a in the domain of both functions.
- ☐ True: $dF/dx > 0$ on every interval where $G > 0$.
- ☐ False: $dG/dx < 0$ on every interval where $F < 0$.