1. The following two problems are not related.

   (a) (10 pts) Find all values of θ in the interval $[0, 2\pi]$ that satisfy $\sin(2\theta) = \cos\theta$.

   Solution:

   This problem is similar to Written HW 1 #1.

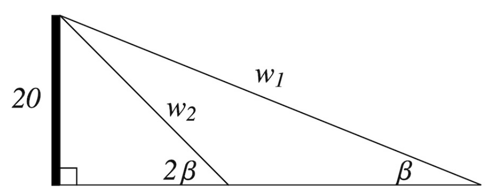
   Use the double angle identity for $\sin(2\theta)$.

$$\begin{aligned} \sin(2\theta) &= \cos\theta \\ 2\sin\theta\cos\theta &= \cos\theta \\ 2\sin\theta\cos\theta - \cos\theta &= 0 \\ (\cos\theta)(2\sin\theta - 1) &= 0 \end{aligned}$$

   The equation is satisfied if $\cos\theta = 0$ at $\theta = \boxed{\frac{\pi}{2}, \frac{3\pi}{2}}$, or $\sin\theta = \frac{1}{2}$ at $\theta = \boxed{\frac{\pi}{6}, \frac{5\pi}{6}}$.

   (b) (14 pts) Two wires of different lengths are stretched tightly from the top of a 20-ft pole to the ground. The longer wire is w_1 feet long and forms an angle of β radians with the ground. The shorter wire is w_2 feet long and forms an angle of 2β radians with the ground. Given $\tan\beta = \frac{2}{5}$, find the following values and fully simplify your answers. (*Hint:* A double angle identity might be useful here.)

   i. w_1 ii. $\cos\beta$ iii. w_2



   Solution:

   Let x be the base length of the large right triangle shown.

   i. First use $\tan\beta = \frac{2}{5}$ to find x.

$$\tan\beta = \frac{\text{opp}}{\text{adj}} = \frac{20}{x} = \frac{2}{5} \implies x = 50.$$

   Then by the Pythagorean Theorem,

$$w_1^2 = 20^2 + x^2 = 20^2 + 50^2 = 2900 \implies w_1 = \boxed{\sqrt{2900}} = \boxed{10\sqrt{29}}.$$

   ii. $\cos\beta = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{x}{w_1} = \dfrac{50}{10\sqrt{29}} = \boxed{\dfrac{5}{\sqrt{29}}}$

   iii. Given that $\sin\beta = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{20}{10\sqrt{29}} = \dfrac{2}{\sqrt{29}}$, we can use the double angle identity for $\sin(2\beta)$ to find that

$$\sin(2\beta) = 2\sin\beta\cos\beta = 2\cdot\frac{2}{\sqrt{29}}\cdot\frac{5}{\sqrt{29}} = \frac{20}{29},$$

then solve for w_2:

$$\sin(2\beta) = \frac{\text{opp}}{\text{hyp}} = \frac{20}{w_2} = \frac{20}{29} \implies w_2 = \boxed{29}.$$

Alternative Solution: Use a double angle identity to find $\cos(2\beta)$:

$$\cos(2\beta) = 2\cos^2\beta - 1 = 2\left(\frac{5}{\sqrt{29}}\right)^2 - 1 = \frac{50}{29} - 1 = \frac{21}{29}.$$

In the small right triangle, let the length of the base equal $21k$ and let $w_2 = 29k$. Applying the Pythagorean Theorem gives

$$20^2 + (21k)^2 = (29k)^2 \implies 400 + 441k^2 = 841k^2 \implies k = 1.$$

Therefore $w_2 = 29$.

2. (18 pts) The function $f(x)$ is defined on the interval $[-5, 0) \cup (0, 5]$. Part of the formula for $f(x)$ is shown below:

$$f(x) = \begin{cases} -2x - 6 & \text{if } -5 \le x < -2 \\ -2 & \text{if } -2 \le x < 0 \\ \dots \end{cases}$$

(a) For each expression, find the value if it exists. Otherwise write DNE. No justification is necessary.

i. $(f \circ f)(-3)$

ii. $\lim_{x\to -2} f(x)$

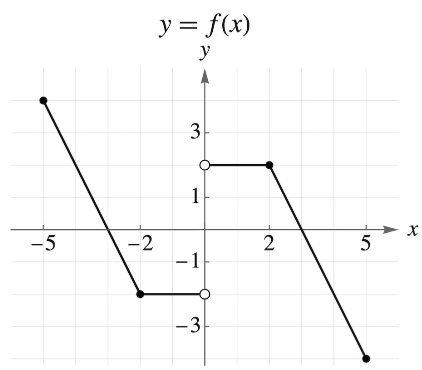
(b) Suppose $f(x)$ is odd. Write out the missing part of the formula for $f(x)$ on $(0, 5]$.

(c) For each expression, find the value if it exists. Otherwise write DNE. No justification is necessary.

i. $\lim_{x\to 5^-} f(x)$

ii. $\lim_{x\to 1/2} f\left(\frac{10}{x+2}\right)$

Solution:



(a) i. $(f \circ f)(-3) = f(f(-3)) = f(-2(-3) - 6) = f(0) = \boxed{\text{DNE}}.$

ii. Examine the limits approaching $x = -2$ from the left and right, respectively.

$$\lim_{x\to -2^-} f(x) = -2(-2) - 6 = -2 \qquad \text{and} \qquad \lim_{x\to -2^+} f(x) = -2.$$

The one-sided limit values are equal so $\lim_{x\to -2} f(x) = \boxed{-2}.$

(b) The missing part of the formula can be determined by first graphing $f(x)$ on $[-5,0)$, then applying symmetry about the origin.

$$f(x) = \begin{cases} -2x-6 & \text{if } -5 \le x < -2 \\ -2 & \text{if } -2 \le x < 0 \\ 2 & \text{if } 0 < x \le 2 \\ -2x+6 & \text{if } 2 < x \le 5 \end{cases}$$

Alternative Solution: For the interval $2 < x \le 5$, the formula can be calculated algebraically. Given that $f(x) = -2x-6$ for $-5 \le x < -2$, we have $f(-x) = -2(-x) - 6 = 2x - 6$. Then, because f is odd, for x in $(2,5]$,

$$f(x) = -f(-x) = -(2x-6) = -2x+6.$$

(c) i. $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} (-2x+6) = \boxed{-4}$.

An alternate way to evaluate the limit is to note that $\lim_{x\to -5^+} f(x) = \lim_{x\to -5^+} (-2x-6) = 4$ and f is odd, so $\lim_{x\to 5^-} f(x) = -4$.

ii. $\lim_{x\to 1/2} f\left(\frac{10}{x+2}\right) = f\left(\frac{10}{5/2}\right) = f(4) = \boxed{-2}$ by direct substitution.

3. (20 pts) Evaluate the limit. If it does not exist, explain why. *(Reminder: You may not use L'Hopital's Rule in any solutions on this exam.)*

(a) $\lim_{x\to 0} \frac{7x\sin x}{9x^2 + x\sin x}$

(b) $\lim_{x\to 0} x^{2/3}\cos\left(\frac{8}{x^2}\right)$

Solution:

(a) *This problem is similar to WebAssign HW 1.4 #4.*

Use the limit $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$. Divide numerator and denominator by x^2 (or divide by x twice).

$$\begin{aligned} \lim_{x\to 0} \frac{7x\sin x}{9x^2 + x\sin x} &= \lim_{x\to 0} \frac{7\sin x}{9x + \sin x} \\ &= \lim_{x\to 0} \frac{7 \cdot \frac{\sin x}{x}}{9 + \frac{\sin x}{x}} \\ &= \frac{7\cdot 1}{9+1} = \boxed{\frac{7}{10}} \end{aligned}$$

Alternate Solution: Divide numerator and denominator by $x\sin x$.

$$\lim_{x\to 0} \frac{7}{9\cdot\frac{x}{\sin x} + 1} = \frac{7}{9\cdot 1 + 1} = \frac{7}{10}$$

Alternate Solution:

$$\lim_{x\to 0} \frac{7x\sin x}{9x^2 + x\sin x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{7x}{9x + \sin x}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{7}{9 + \frac{\sin x}{x}}$$

$$= 1 \cdot \frac{7}{9+1} = \boxed{\frac{7}{10}}$$

(b) Use the Squeeze Theorem. For $x \neq 0$,

$$-1 \le \cos\left(\frac{8}{x^2}\right) \le 1$$

and

$$-x^{2/3} \le x^{2/3} \cos\left(\frac{8}{x^2}\right) \le x^{2/3}.$$

Because

$$\lim_{x\to 0} -x^{2/3} = \lim_{x\to 0} x^{2/3} = 0,$$

$\lim_{x\to 0} x^{2/3} \cos\left(\frac{8}{x^2}\right)$ also equals $\boxed{0}$.

4. (18 pts) Consider the function

$$g(x) = \begin{cases} \dfrac{10 - \sqrt{x}}{100 - x} & \text{if } x > 100 \\ \dfrac{x}{c} & \text{if } x \le 100 \end{cases}$$

where c is a nonzero constant.

(a) Evaluate $\lim_{x\to 100^+} g(x)$.

(b) Is there a value for c that will make $g(x)$ continuous at $x = 100$? If so, find c. If not, explain why there is no value. State the definition of continuity and use it to justify your answer.

Solution:

(a)

$$\begin{aligned} \lim_{x\to 100^+} \frac{10 - \sqrt{x}}{100 - x} &= \lim_{x\to 100^+} \frac{10 - \sqrt{x}}{100 - x} \cdot \frac{10 + \sqrt{x}}{10 + \sqrt{x}} \\ &= \lim_{x\to 100^+} \frac{\cancel{100 - x}}{\cancel{(100 - x)}\,(10 + \sqrt{x})} \\ &= \lim_{x\to 100^+} \frac{1}{10 + \sqrt{x}} = \boxed{\frac{1}{20}} \end{aligned}$$

(b) The function $g(x)$ is continuous at $x = 100$ if

$$g(100) = \lim_{x\to 100^-} g(x) = \lim_{x\to 100^+} g(x).$$

The value of $g(100)$ equals the value of the limit of $g(x)$ as $x \to 100^-$:

$$g(100) = \lim_{x\to 100^-} \frac{x}{c} = \frac{100}{c}.$$

The limit of $g(x)$ as $x \to 100^+$, found in part (a), equals $\frac{1}{20}$.
Setting $g(100)$ and the one-sided limits equal gives the value of c:

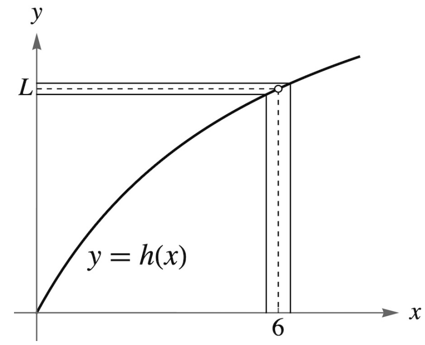
$$\frac{100}{c} = \frac{1}{20} \implies \boxed{c = 2000}.$$

5. (20 pts) Consider the function $h(x) = \dfrac{8x^2 - 48x}{x^2 - 36}$.

(a) Find the domain of $h(x)$. Express your answer in interval notation.

(b) Find the value of $\lim\limits_{x\to 6} h(x)$ or explain why it does not exist.

(c) Shown at right is an illustration of $\lim\limits_{x\to 6} h(x)$ using the precise definition of a limit. It shows that for $x \neq 6$, if x is in the interval $(5.7, 6.3)$, then $h(x)$ is in the interval $(L - 0.11, L + 0.11)$. Identify L, δ, and ϵ if they exist. No explanation is necessary.



Solution:

(a) The function is defined for all real numbers except where the denominator $x^2 - 36$ equals zero at $x = \pm 6$. Therefore the domain of $h(x)$ is $\boxed{(-\infty, -6) \cup (-6, 6) \cup (6, \infty)}$.

(b) $\lim\limits_{x\to 6} h(x) = \lim\limits_{x\to 6} \dfrac{8x\cancel{(x-6)}}{\cancel{(x-6)}(x+6)} = \lim\limits_{x\to 6} \dfrac{8x}{x+6} = \dfrac{8\cdot 6}{6+6} = \boxed{4}$.

(c) In part (b), we found that $\lim\limits_{x\to 6} h(x) = 4$, so $\boxed{L = 4}$. The precise definition of this limit says that for every $\epsilon > 0$, there is a corresponding $\delta > 0$ such that

$$0 < |x - 6| < \delta \implies |h(x) - 4| < \epsilon.$$

We are given the interval $5.7 < x < 6.3$, which can be written as $|x - 6| < 0.3$. For any value of x in the interval, the value of $h(x)$ will satisfy $4 - 0.11 < h(x) < 4 + 0.11$, which can be written as $|h(x) - 4| < 0.11$. Therefore $\boxed{\delta = 0.3}$ and $\boxed{\epsilon = 0.11}$.