1. (30pts) Let $f(x) = x\,e^{-x}$. You may use the fact that the kth derivative of f is

$$f^{(k)}(x) = (-1)^k e^{-x}(x-k), \qquad k \geq 1.$$

(a) (15pts) Find the Maclaurin series for $f(x)$, written in sigma notation.

Solution: There are two valid approaches; either earns full credit.

Method 1: from the definition. Using the given formula, $f^{(k)}(0) = (-1)^k e^0(0-k) = (-1)^{k-1}k$ for $k \geq 1$, and $f(0) = 0$. So

$$f'(0) = (-1)^0(1) = 1, \qquad f''(0) = (-1)^1(2) = -2,$$
$$f'''(0) = (-1)^2(3) = 3, \qquad f^{(4)}(0) = (-1)^3(4) = -4.$$

Using the Maclaurin definition $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ (the $n = 0$ term is 0),

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n!}x^n = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!}x^n} = x - x^2 + \frac{x^3}{2} - \cdots$$

Method 2: from a known series. Substitute $-x$ into $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$ and multiply by x:

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}, \qquad xe^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!}.$$

Re-indexing with $m = n+1$ gives $\sum_{m=1}^{\infty} \frac{(-1)^{m-1}x^m}{(m-1)!}$, the same series.

(b) (5pts) Write $T_3(x)$, the third-order Maclaurin polynomial for f.

Solution: Taking the $n = 1, 2, 3$ terms of the series,

$$T_3(x) = x - x^2 + \frac{x^3}{2}.$$

(c) (10pts) Use Taylor's formula to find an error bound when $T_3(x)$ is used to approximate $f(x)$ on the interval $0 \leq x \leq 1$.

Solution: By Taylor's formula, the remainder is

$$R_3(x) = \frac{f^{(4)}(z)}{4!}x^4$$

for some z between 0 and x. From the given formula, $f^{(4)}(x) = e^{-x}(x-4)$, so for $0 \leq z \leq 1$,

$$\left|f^{(4)}(z)\right| = e^{-z}\,|z-4| \leq 1 \cdot 4 = 4$$

(since $e^{-z} \leq 1$ and $|z-4| \leq 4$ on $[0,1]$). On $0 \leq x \leq 1$ we also have $x^4 \leq 1$. Therefore

$$|R_3(x)| \leq \frac{4}{4!} = \frac{4}{24} = \frac{1}{6}.$$

2. (20pts) Consider the parametric curve

$$x = t^2, \qquad y = t^3 - t, \qquad t \geq 0.$$

(a) (10pts) Find an equation for the line tangent to the curve at the point where $t = 1$.

Solution: Differentiate: $\dfrac{dx}{dt} = 2t$ and $\dfrac{dy}{dt} = 3t^2 - 1$, so

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 1}{2t}.$$

At $t = 1$ the slope is

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{3(1)^2 - 1}{2(1)} = \frac{2}{2} = 1.$$

The point of tangency is $(x, y) = \big(1^2,\ 1^3 - 1\big) = (1, 0)$. Therefore the tangent line is

$$y - 0 = 1\,(x - 1) \quad \Longrightarrow \quad \boxed{y = x - 1}.$$

(b) (10pts) The curve begins at the origin when $t = 0$. **Set up, but do not evaluate**, an integral that gives the length of the curve from the origin until the curve **next crosses the x-axis**.

Solution: Find the bounds. The curve is on the x-axis when $y = 0$:

$$t^3 - t = 0 \implies t\big(t^2 - 1\big) = 0 \implies t = 0,\ \pm 1.$$

Since $t \geq 0$, we discard $t = -1$. The curve starts at the origin at $t = 0$, and the next value of t giving $y = 0$ is $t = 1$. (For $0 < t < 1$ we have $y < 0$, so the curve dips below the axis and returns.) Hence the bounds are $t = 0$ to $t = 1$.

Set up the integral. The arc length element is

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{(2t)^2 + \big(3t^2 - 1\big)^2}\,dt.$$

Expanding the integrand,

$$(2t)^2 + \big(3t^2 - 1\big)^2 = 4t^2 + 9t^4 - 6t^2 + 1 = 9t^4 - 2t^2 + 1.$$

Therefore

$$\boxed{L = \int_0^1 \sqrt{9t^4 - 2t^2 + 1}\,dt}$$

3. (30pts) The following problems are not related.

(a) (18pts) Consider the **cardioid** $r = 1 + \cos\theta$. Find all values of θ in $[0, 2\pi)$ at which the cardioid has a horizontal tangent line. If the slope is indeterminate, simply state the value of θ this occurs at.

Solution: Write $y = r\sin\theta = (1 + \cos\theta)\sin\theta = \sin\theta + \frac{1}{2}\sin(2\theta)$. Then

$$\frac{dy}{d\theta} = \cos\theta + \cos(2\theta) = 2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1).$$

Setting $\dfrac{dy}{d\theta} = 0$ gives $\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$, i.e. $\theta = \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}$. We need to check $\dfrac{dx}{d\theta} \neq 0$. Since $x = \cos\theta + \cos^2\theta$,

$$\frac{dx}{d\theta} = -\sin\theta - \sin(2\theta) = -\sin\theta\,(1 + 2\cos\theta).$$

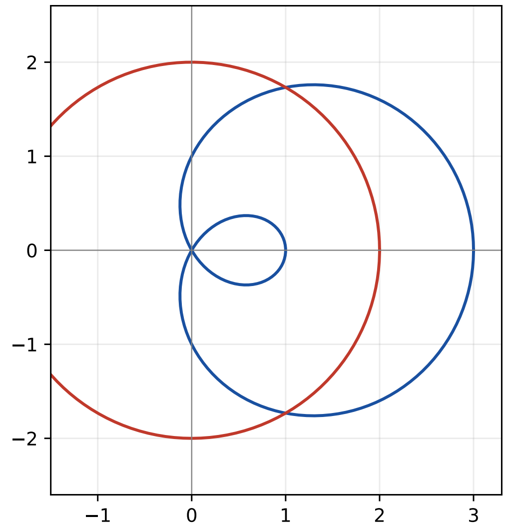
At $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$ this is nonzero, so the horizontal tangents are

$$\boxed{\theta = \frac{\pi}{3} \quad \text{and} \quad \theta = \frac{5\pi}{3}.}$$

At $\theta = \pi$ we have $\dfrac{dy}{d\theta} = 0$ *and* $\dfrac{dx}{d\theta} = -\sin\pi\,(1 + 2\cos\pi) = 0$, so $\dfrac{dy}{dx}$ is the indeterminate form $\frac{0}{0}$. This is the cusp of the cardioid, at the origin:

$$\boxed{\theta = \pi.}$$

(b) (12pts) **Set up, but do not evaluate**, an integral that gives the area of the region that lies **inside** $r = 1 + 2\cos\theta$ and **outside** $r = 2$.



Solution: First find where the curves meet:

$$1 + 2\cos\theta = 2 \implies \cos\theta = \tfrac{1}{2} \implies \theta = \pm\tfrac{\pi}{3}.$$

For $-\frac{\pi}{3} < \theta < \frac{\pi}{3}$ we have $1 + 2\cos\theta > 2$, so the limaçon is the outer boundary and the circle is the inner boundary. Therefore

$$\boxed{A = \int_{-\pi/3}^{\pi/3} \frac{1}{2}\left[(1 + 2\cos\theta)^2 - 2^2\right] d\theta}$$

Equivalently, using symmetry about the x-axis,

$$A = 2 \cdot \frac{1}{2}\int_0^{\pi/3} \left[(1 + 2\cos\theta)^2 - 2^2\right] d\theta = \int_0^{\pi/3} \left[(1 + 2\cos\theta)^2 - 4\right] d\theta.$$

(For reference, the value is $A = \frac{5\sqrt{3}}{2} - \frac{\pi}{3}$.)

4. (45pts) Review soup! **Answer <u>THREE</u> of the following six problems.** Each is worth 15 points. **You may attempt as many as you like and your highest three will count towards your score**.

(a) Evaluate $\displaystyle\int_0^{\pi/2} e^x \cos x\, dx$.

Solution: First find the antiderivative by integration by parts twice (a "boomerang"). Let $I = \displaystyle\int e^x \cos x\, dx$. Take $u = \cos x$, $dv = e^x\, dx$, so $du = -\sin x\, dx$, $v = e^x$:

$$I = e^x \cos x + \int e^x \sin x\, dx.$$

For the remaining integral, take $u = \sin x$, $dv = e^x\,dx$, giving $\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx = e^x \sin x - I$. Substitute back:

$$I = e^x \cos x + e^x \sin x - I \implies 2I = e^x(\cos x + \sin x) \implies I = \frac{e^x(\cos x + \sin x)}{2}.$$

Now evaluate on $[0, \frac{\pi}{2}]$:

$$\int_0^{\pi/2} e^x \cos x\,dx = \left[\frac{e^x(\cos x + \sin x)}{2}\right]_0^{\pi/2} = \frac{e^{\pi/2}(0+1)}{2} - \frac{e^0(1+0)}{2} = \boxed{\frac{e^{\pi/2}-1}{2}}$$

(b) Solve the initial value problem for y as a function of x:

$$\frac{dy}{dx} = y\,\sec^4 x, \qquad y(0) = 1.$$

Solution: Separate variables:

$$\int \frac{dy}{y} = \int \sec^4 x\,dx.$$

For the right-hand integral, save a factor of $\sec^2 x$ and convert the rest using $\sec^2 x = 1 + \tan^2 x$:

$$\int \sec^4 x\,dx = \int (1 + \tan^2 x)\sec^2 x\,dx.$$

Substitute $u = \tan x$, $du = \sec^2 x\,dx$:

$$\int (1 + u^2)\,du = u + \frac{u^3}{3} + C = \tan x + \frac{1}{3}\tan^3 x + C.$$

Therefore $\ln|y| = \tan x + \frac{1}{3}\tan^3 x + C$. Apply $y(0) = 1$: since $\tan 0 = 0$, $\ln 1 = 0 = C$, so $C = 0$. Exponentiating,

$$\boxed{y = \exp\left(\tan x + \tfrac{1}{3}\tan^3 x\right).}$$

(c) Let $\mathcal{R}$ be the region bounded by the line $y = 2x$ and the parabola $y = x^2$. Find the volume of the solid generated by rotating $\mathcal{R}$ about the line $x = 3$.

Solution: The curves meet where $2x = x^2$, i.e. $x^2 - 2x = 0$, giving $x = 0$ and $x = 2$; on $[0, 2]$ the line lies above the parabola ($2x \geq x^2$).

The axis of rotation $x = 3$ is vertical, so the **cylindrical shell** method (integrating in x) is most direct. A shell at position x has radius $r = 3 - x$ (distance to the axis) and height $h = 2x - x^2$:

$$V = \int_0^2 2\pi(3-x)\,(2x - x^2)\,dx = 2\pi \int_0^2 \left(6x - 3x^2 - 2x^2 + x^3\right)dx = 2\pi \int_0^2 \left(6x - 5x^2 + x^3\right)dx.$$

Evaluate:

$$V = 2\pi\left[3x^2 - \frac{5x^3}{3} + \frac{x^4}{4}\right]_0^2 = 2\pi\left(12 - \frac{40}{3} + 4\right) = 2\pi\left(16 - \frac{40}{3}\right) = 2\pi \cdot \frac{8}{3} = \boxed{\frac{16\pi}{3}}.$$

(d) Determine whether the improper integral converges or diverges. If it converges, find its value:

$$\int_1^\infty \frac{\ln x}{x^2}\,dx.$$

Solution: Integration by parts with $u = \ln x$, $dv = x^{-2}\,dx$, so $du = \frac{1}{x}\,dx$, $v = -x^{-1}$:

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

Then

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left(-\frac{\ln t}{t} - \frac{1}{t}\right) - (0-1).$$

Since $\lim_{t\to\infty} \frac{\ln t}{t} = 0$ (e.g. by L'Hôpital) and $\lim_{t\to\infty} \frac{1}{t} = 0$, the integral **converges** to

$$0 - (0-1) = \boxed{1}.$$

(e) Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Fully justify your answer.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{2n}}.$$

Solution: Not absolutely convergent. The absolute series is

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{2n}} = \frac{1}{2^{1/3}} \sum_{n=1}^{\infty} \frac{1}{n^{1/3}},$$

a constant multiple of a p-series with $p = \frac{1}{3} \le 1$, which diverges.

Convergence (AST). The series alternates with $b_n = \dfrac{1}{\sqrt[3]{2n}} = \dfrac{1}{(2n)^{1/3}} > 0$. Since $2n$ increases with n, b_n is decreasing, and $\lim_{n\to\infty} b_n = 0$ (because $(2n)^{1/3} \to \infty$). By the Alternating Series Test, the series converges.

Since the series converges but its absolute series diverges, it is **conditionally convergent**.

(f) Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n\,2^n}.$$

Solution: Ratio Test with $a_n = \dfrac{(x-3)^n}{n2^n}$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-3|^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{|x-3|^n} = \frac{n}{n+1} \cdot \frac{|x-3|}{2} \xrightarrow[n\to\infty]{} \frac{|x-3|}{2}.$$

Convergence for $\frac{|x-3|}{2} < 1$, i.e. $|x-3| < 2$, so $R = 2$ and $1 < x < 5$. Check endpoints:

At $x = 5$**:** $\sum \dfrac{2^n}{n2^n} = \sum \dfrac{1}{n}$, the harmonic series, which diverges.

At $x = 1$**:** $\sum \dfrac{(-2)^n}{n2^n} = \sum \dfrac{(-1)^n}{n}$, the alternating harmonic series, which converges.

Therefore the interval of convergence is $\boxed{[1,5)}$ with $R = 2$.