Write your name below. This exam is worth 100 points and has 6 problems. You must show your work to receive full credit on the problems. You are allowed to use both sides of an 8.5x11 page of notes. You cannot collaborate on the exam or seek outside help, nor can you use the recorded lectures, a calculator, any computational software, or material you find online.

Name:

1. (24 points, 8 each) In each of the following problems, either show that the statement is always true or provide a counterexample that shows it is false.

   (a) If A is a symmetric matrix and the vector $\mathbf{b}$ is orthogonal to $\ker A$, then the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.

   (b) If the linear function $L : \mathbb{R}^2 \to \mathbb{R}^2$ has matrix representation A, so $L(\mathbf{x}) = A\mathbf{x}$, then the function represented by the matrix A^2 is not linear.

   (c) If the square symmetric matrix K is not positive definite, then the quadratic form $q(\mathbf{x}) = \mathbf{x}^T K\mathbf{x}$ does not have a minimum on the unit sphere.

Solution: (a) **True.** Since A is symmetric, $\ker A = \operatorname{coker} A$, so if $\mathbf{b}$ is orthogonal to the kernel, it is also orthogonal to the cokernel. The Fredholm alternative then tells us there is at least one solution to $A\mathbf{x} = \mathbf{b}$.

(b) **False.** The function is $L(\mathbf{x}) = A^2\mathbf{x}$, which is still linear, since

$$L(\alpha\mathbf{x} + \beta\mathbf{y}) = A^2(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha A^2\mathbf{x} + \beta A^2\mathbf{y} = \alpha L(\mathbf{x}) + \beta L(\mathbf{y})$$

(c) **False.** A counterexample is $K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which has a minimum value of the smallest eigenvalue, which is -1:

$$q(\mathbf{e}_2) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1$$

2. (16 points) Let $A = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$

   (a) (4 points) What are the eigenvalues of A?

   (b) (4 points) What are the eigenvectors of A?

   (c) (8 points) Find e^{At}. Make sure to simplify your answer.

Solution: (a) We calculate the roots of A's characteristic polynomial:

$$\begin{aligned} \det\left(A - \lambda I\right) &= \left| \begin{pmatrix} 3-\lambda & 2 \\ 2 & -\lambda \end{pmatrix} \right| \\ &= (3-\lambda)(-\lambda) - 2^2 \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda - 4)(\lambda + 1) \end{aligned}$$

So we have $\lambda = 4$ and $\lambda = -1$.

(b) We calculate the kernels of $A - \lambda I$ for each value of λ to find the eigenvectors:

$$A - (-1)I = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \xrightarrow[R_2 \to R_2 - (1/2)R_1]{R_1 \to (1/2)R_1} \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{v}_{-1} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$A - 4I = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \xrightarrow[R_2 \to R_2 + 2R_1]{} \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{v}_4 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

(c) Once we have diagonalized the matrix so $A = S\Lambda S^{-1}$, we will have

$$e^{At} = e^{S\Lambda S^{-1}t} = Se^{\Lambda t}S^{-1}$$

We already have the two matrices we need. Using the eigenvalues and eigenvectors we have

$$\Lambda = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

Now calculate S^{-1}. Using the 2×2 formula gives:

$$S^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

So we have

$$\begin{aligned} e^{At} &= \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{4t} \end{pmatrix} \frac{1}{5} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & -2e^{-t} \\ 2e^{4t} & e^{4t} \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} e^{-t} + 4e^{4t} & -2e^{-t} + 2e^{4t} \\ -2e^{-t} + 2e^{4t} & 4e^{-t} + e^{4t} \end{pmatrix} \end{aligned}$$

3. (13 points) Let A be an incomplete matrix with the Jordan Factorization of

$$A = S \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix} S^{-1}$$

where S has columns $\mathbf{s}_1$, $\mathbf{s}_2$, etc:

$$S = \begin{pmatrix} | & | & | & | \\ \mathbf{s}_1 & \mathbf{s}_2 & \dots & \mathbf{s}_6 \\ | & | & | & | \end{pmatrix}$$

(a) (2 points) What are the **complete** eigenvalues of A?

(b) (2 points) Which $\mathbf{s}_i$ are the **ordinary** eigenvectors of A?

(c) (3 points) Is $(\mathbf{s}_1 + \mathbf{s}_2)$ an ordinary eigenvector, a generalized eigenvector, or neither?

(d) (3 points) Is $(\mathbf{s}_1 + \mathbf{s}_3)$ an ordinary eigenvector, a generalized eigenvector, or neither?

(e) (3 points) Is $(\mathbf{s}_1 + \mathbf{s}_5)$ an ordinary eigenvector, a generalized eigenvector, or neither?

Solution: (a) The only eigenvalue where the algebraic multiplicity equals the geometric multiplicity is $\lambda = -1$. It has a single 1×1 Jordan block.

(b) There are four total Jordan blocks, each representing a Jordan chain. The chains start with the ordinary eigenvectors, which are $\mathbf{s}_1$, $\mathbf{s}_3$, $\mathbf{s}_4$, and $\mathbf{s}_5$.

(c) This is a generalized eigenvector. We determine this by using the general eigenvector equation:

$$\begin{aligned} A(\mathbf{s}_1 + \mathbf{s}_2) &= A\mathbf{s}_1 + A\mathbf{s}_2 = 2\mathbf{s}_1 + 2\mathbf{s}_2 + \mathbf{s}_1 \\ &= 2(\mathbf{s}_1 + \mathbf{s}_2) + \mathbf{s}_1 \end{aligned}$$

(d) This is an ordinary eigenvector. We determine this by using the general eigenvector equation:

$$\begin{aligned} A(\mathbf{s}_1 + \mathbf{s}_3) &= A\mathbf{s}_1 + A\mathbf{s}_3 = 2\mathbf{s}_1 + 2\mathbf{s}_3 \\ &= 2(\mathbf{s}_1 + \mathbf{s}_3) \end{aligned}$$

(e) This is neither, since $\mathbf{s}_1$ and $\mathbf{s}_5$ have different eigenvalues:

$$A(\mathbf{s}_1 + \mathbf{s}_5) = A\mathbf{s}_1 + A\mathbf{s}_5 = 2\mathbf{s}_1 + 3\mathbf{s}_5$$

4. (16 points) The matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 2 & 0 \end{pmatrix}$ has singular values $\sigma_1 = 3$ and $\sigma_2 = 2$.

   (a) (8 points) What are the singular vectors of A?

   (b) (4 points) Find the best rank 1 approximation of A.

   (c) (4 points) Find the pseudoinverse of A. You may leave your answer as a product of matrices.

Solution: (a) We first need to calculate $A^T A$:

$$A^T A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 2 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$$

The singular vectors are the eigenvectors of $A^T A$, and the eigenvalues of $A^T A$ are the squares of the singular values:
$\sigma_1 = 3 \to \lambda_1 = 9$:

$$\left(A^T A - 9I\right) = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \to \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$$
$$\mathbf{q}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\sigma_2 = 2 \to \lambda_2 = 4$:

$$\left(A^T A - 4I\right) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$
$$\mathbf{q}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Where we have normalized these singular vectors.

(b) For this approximation we need to calculate the first column of P:

$$\mathbf{p}_1 = \frac{A\mathbf{q}_1}{\sigma_1} = \frac{1}{3} \begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 2 & 0 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$= \frac{1}{3\sqrt{5}} \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix}$$

As we have already normalized $\mathbf{q}_1$, this $\mathbf{p}_1$ is a unit vector. The best rank 1 approximation is then

$$A_1 = \sigma_1 \mathbf{p}_1 \mathbf{q}_1^T = 3 \frac{1}{3\sqrt{5}} \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \end{pmatrix}$$
$$= \frac{1}{5} \begin{pmatrix} 5 & 10 \\ -4 & -8 \\ 2 & 4 \end{pmatrix}$$

(c) For the pseudoinverse, we need the entire P matrix, so we calculate the second column:

$$\begin{aligned}\mathbf{p}_2 = \frac{A\mathbf{q}_2}{\sigma_2} &= \frac{1}{2}\begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 2 & 0 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \\ &= \frac{1}{2\sqrt{5}}\begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix} = \frac{1}{3\sqrt{5}}\begin{pmatrix} 0 \\ -3 \\ -6 \end{pmatrix}\end{aligned}$$

We now have all of the matrices we need and can calculate the pseudoinverse:

$$\begin{aligned}A^+ &= Q\Sigma^{-1}P^T \\ &= \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix}\frac{1}{3\sqrt{5}}\begin{pmatrix} 5 & -4 & 2 \\ 0 & -3 & -6 \end{pmatrix}\end{aligned}$$

5. (16 points) A linear map $L : \mathbb{R}^3 \to \mathbb{R}^2$ has matrix representation $A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 2 & 2 \end{pmatrix}$.

   (a) (3 points) What is the canonical form of A?

   (b) (8 points) If you were to perform a change of basis on A such that it is placed in canonical form, what would your $\mathbb{R}^3$ basis be?

   (c) (5 points) What would your $\mathbb{R}^2$ basis be?

Solution: (a) Calculating the RREF, we find that A is rank 2:

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & 2 & 2 \end{pmatrix} \xrightarrow[R_2 \to R_2 - R_1]{} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 3 & 3 \end{pmatrix}$$
$$\xrightarrow[R_3 \to \frac{1}{3} R_3]{R_1 \to R_1 + \frac{1}{3} R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

So the canonical form of A is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(b) The first two basis vectors are a basis for the coimage, so we take the nonzero rows of the RREF for this basis:

$$\mathbf{s}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{s}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

The last vectors must be chosen to be a basis for the kernel, so we use the RREF to calculate it. Our free variable is $\mathbf{x}_3$:

$$\mathbf{s}_3 = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\to x = 0 \ , \ y = -1$$
$$\to \mathbf{s}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

and so our $\mathbb{R}^3$ basis is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} , \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

(c) For the $\mathbb{R}^2$ basis, we first find $L(\mathbf{s}_1)$ and $L(\mathbf{s}_2)$:

$$L(\mathbf{s}_1) = A\mathbf{s}_1 = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$L(\mathbf{s}_2) = A\mathbf{s}_2 = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

and so our $\mathbb{R}^2$ basis is

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \begin{pmatrix} 0 \\ 4 \end{pmatrix} \right\}$$

6. (15 points) Let $p(\mathbf{x}) = x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + x_3^2 - 2\mathbf{x}^T\mathbf{f}$ where $\mathbf{f} \in \mathbb{R}^3$.

   (a) (7 points) Does $p(\mathbf{x})$ always have a minimizer (or minimizers) regardless of what $\mathbf{f}$ is? Show that it does or find an $\mathbf{f}$ where it doesn't.

   (b) (6 points) Find all of the minimizers of $p(\mathbf{x})$ when $\mathbf{f} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

   (c) (2 points) For the $\mathbf{f}$ from part b, what is the minimum value of $p(\mathbf{x})$?

Solution: (a) We first find the K matrix such that $p(\mathbf{x}) = \mathbf{x}^T K\mathbf{x} - 2\mathbf{x}^T\mathbf{f}$ by reading off the coefficients:

$$K = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

This matrix is positive semidefinite:

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

This implies that $p(\mathbf{x})$ will not have minimizers unles the equation $K\mathbf{x} = \mathbf{f}$ has solutions, in other words when $\mathbf{f} \in \operatorname{img} K$. So no, $p(\mathbf{x})$ not always have minima. An example of an $\mathbf{f}$ vector where $p(\mathbf{x})$ doesn't is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ since $K\mathbf{x} = \mathbf{f}$ is inconsistent:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ -1 & 2 & -1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

(b) The minimizers are the solutions to $K\mathbf{x} = \mathbf{f}$, so we solve:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ -1 & 2 & -1 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right) \xrightarrow{R_2 \to R_2 + R_1} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -2 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So our minimizers are, remembering to add a general member of the kernel:

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} t$$

(c) We calculate $p(\mathbf{x})$ for our $\mathbf{x}$ from part b. Since every value of t gives a minimizer, we can choose $t = 0$. Using $K\mathbf{x} = \mathbf{f}$ gives

$$
\begin{aligned}
p(\mathbf{x}) &= \mathbf{x}^T K \mathbf{x} - 2\mathbf{x}^T \mathbf{f} = \mathbf{x}^T \mathbf{f} - 2\mathbf{x}^T \mathbf{f} = -\mathbf{x}^T \mathbf{f} \\
&= -\begin{pmatrix} 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \\
&= -5
\end{aligned}
$$