Write your name below. This exam is worth 100 points and has 6 problems. You must show your work to receive full credit on the problems. You are allowed to use both sides of an 8.5x11 page of notes. You cannot collaborate on the exam or seek outside help, nor can you use the recorded lectures, a calculator, any computational software, or material you find online.

Name:

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1. (24 points, 8 each) In each of the following problems, either show that the statement is always true or provide a counterexample that shows it is false.

   (a) If A is a symmetric matrix and the vector $\mathbf{b}$ is orthogonal to $\ker A$, then the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.

   (b) If the linear function $L : \mathbb{R}^2 \to \mathbb{R}^2$ has matrix representation A, so $L(\mathbf{x}) = A\mathbf{x}$, then the function represented by the matrix A^2 is not linear.

   (c) If the square symmetric matrix K is not positive definite, then the quadratic form $q(\mathbf{x}) = \mathbf{x}^T K\mathbf{x}$ does not have a minimum on the unit sphere.

2. (16 points) Let $A = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$

   (a) (4 points) What are the eigenvalues of A?

   (b) (4 points) What are the eigenvectors of A?

   (c) (8 points) Find e^{At}. Make sure to simplify your answer.

3. (13 points) Let A be an incomplete matrix with the Jordan Factorization of

$$A = S \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix} S^{-1}$$

   where S has columns $\mathbf{s}_1$, $\mathbf{s}_2$, etc:

$$S = \begin{pmatrix} | & | & & | \\ \mathbf{s}_1 & \mathbf{s}_2 & \dots & \mathbf{s}_6 \\ | & | & & | \end{pmatrix}$$

   (a) (2 points) What are the **complete** eigenvalues of A?

   (b) (2 points) Which $\mathbf{s}_i$ are the **ordinary** eigenvectors of A?

   (c) (3 points) Is $(\mathbf{s}_1 + \mathbf{s}_2)$ an ordinary eigenvector, a generalized eigenvector, or neither?

   (d) (3 points) Is $(\mathbf{s}_1 + \mathbf{s}_3)$ an ordinary eigenvector, a generalized eigenvector, or neither?

(e) (3 points) Is $(\mathbf{s}_1 + \mathbf{s}_5)$ an ordinary eigenvector, a generalized eigenvector, or neither?

4. (16 points) The matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 2 & 0 \end{pmatrix}$ has singular values $\sigma_1 = 3$ and $\sigma_2 = 2$.

   (a) (8 points) What are the singular vectors of A?

   (b) (4 points) Find the best rank 1 approximation of A.

   (c) (4 points) Find the pseudoinverse of A. You may leave your answer as a product of matrices.

5. (16 points) A linear map $L : \mathbb{R}^3 \to \mathbb{R}^2$ has matrix representation $A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 2 & 2 \end{pmatrix}$.

   (a) (3 points) What is the canonical form of A?

   (b) (8 points) If you were to perform a change of basis on A such that it is placed in canonical form, what would your $\mathbb{R}^3$ basis be?

   (c) (5 points) What would your $\mathbb{R}^2$ basis be?

6. (15 points) Let $p(\mathbf{x}) = x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + x_3^2 - 2\mathbf{x}^T\mathbf{f}$ where $\mathbf{f} \in \mathbb{R}^3$.

   (a) (7 points) Does $p(\mathbf{x})$ always have a minimizer (or minimizers) regardless of what $\mathbf{f}$ is? Show that it does or find an $\mathbf{f}$ where it doesn't.

   (b) (6 points) Find all of the minimizers of $p(\mathbf{x})$ when $\mathbf{f} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

   (c) (2 points) For the $\mathbf{f}$ from part b, what is the minimum value of $p(\mathbf{x})$?