1. (10 pts) Find the minimum vertical distance between the functions $f(x) = e^{2x}$ and $g(x) = x$
   Solution:
   The vertical distance is given by the difference in y coordinates.

$$h(x) = f(x) - g(x) = e^{2x} - x$$

We find the critical number of this function

$$\begin{aligned} h'(x) &= 0 \\ 2e^{2x} - 1 &= 0 \\ e^{2x} &= \frac{1}{2} \\ x &= \frac{1}{2}\ln\left(\frac{1}{2}\right) \\ &= -\frac{\ln 2}{2} \end{aligned}$$

Now, we perform the 2nd derivative test

$$\begin{aligned} h''(x) &= 4e^{2x} \\ h''\left(-\frac{\ln 2}{2}\right) &= 4e^{-\ln 2} \\ &= 4e^{\ln 2^{-1}} \\ &= 2 > 0 \end{aligned}$$

Hence $x = -\dfrac{\ln 2}{2}$ is a local minimum. The minimum distance is thus given by
$h\left(-\dfrac{\ln 2}{2}\right) = e^{-\ln 2} + \dfrac{\ln 2}{2} = \boxed{\dfrac{1}{2} + \dfrac{\ln 2}{2}}$

2. Differentiate the following with respect to x

   (a) (10 pts) $y = \left(x + e^{\arctan(x)}\right)^2$
   Solution:
   We apply the chain rule
   $y' = 2\left(x + e^{\arctan(x)}\right)\left(1 + e^{\arctan(x)}\left(\dfrac{1}{1+x^2}\right)\right) = \boxed{2\left(x + e^{\arctan(x)}\right)\left(1 + \dfrac{e^{\arctan(x)}}{1+x^2}\right)}$

   (b) (10 pts) $y = x^{\cosh x}$

Solution:
We apply Logarithmic differentiation

$$\begin{aligned} \ln y &= \cosh x \ln x \\ \frac{1}{y}y' &= \cosh x \cdot \frac{1}{x} + \sinh x \ln x \\ y' &= y\left(\frac{\cosh x}{x} + \sinh x \ln x\right) \\ &= \boxed{x^{\cosh x}\left(\frac{\cosh x}{x} + \sinh x \ln x\right)} \end{aligned}$$

3. (10 pts) Find the range of the following function by finding its inverse

$$f(x) = \frac{2x+5}{3x-1}$$

Solution:
We find the inverse by calling $y = \dfrac{2x+5}{3x-1}$, swapping x and y, and solving for y

$$\begin{aligned} x &= \frac{2y+5}{3y-1} \\ 3xy - x &= 2y + 5 \\ y(3x-2) &= x + 5 \\ y &= \frac{x+5}{(3x-2)} \end{aligned}$$

Hence $f^{-1}(x) = \dfrac{x+5}{(3x-2)}$. And the range of f is the domain of f^{-1} which is

$\boxed{\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \infty\right)}$

4. Evaluate the following limits

(a) (10 pts) $\lim\limits_{x\to\infty} e^{-x}x^2$

Solution:
Using L'Hospital's rule

$$\begin{aligned} \lim_{x\to\infty} e^{-x}x^2 &= \lim_{x\to\infty} \frac{x^2}{e^x} \left(\frac{0}{0}\right) \\ &= \lim_{x\to\infty} \frac{2x}{e^x} \left(\frac{0}{0}\right) \\ &= \lim_{x\to\infty} \frac{2}{e^x} \\ &= \boxed{0} \end{aligned}$$

(b) (10 pts) $\lim\limits_{n\to\infty} \left(1 + \dfrac{2}{n}\right)^n$

Solution:

$$\begin{aligned} y &= \lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^n \\ \ln y &= \ln\left(\lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^n\right) \\ &= \lim_{n\to\infty} \ln\left(1 + \frac{2}{n}\right)^n \quad \text{using limit law} \\ &= \lim_{n\to\infty} n \ln\left(1 + \frac{2}{n}\right) \\ &= \lim_{n\to\infty} \frac{\ln\left(1 + \frac{2}{n}\right)}{\left(\frac{1}{n}\right)} \left(\frac{0}{0}\right) \\ &= \lim_{n\to\infty} \frac{\frac{1}{\left(1+\frac{2}{n}\right)}\left(-\frac{2}{n^2}\right)}{-\frac{1}{n^2}} \\ &= \lim_{n\to\infty} \frac{2}{1+\frac{2}{n}} \\ &= 2 \end{aligned}$$

Hence, $y = e^2$, i.e, $\boxed{\lim\limits_{n\to\infty} \left(1 + \frac{2}{n}\right)^n = e^2}$

5. Evaluate the following integrals

(a) (10 pts) $\int \ln(\cosh x) \tanh x \, dx$

Solution:

We employ the substitution

$u = \ln(\cosh x) \Rightarrow du = \frac{1}{\cosh x}(\sinh x)dx = \tanh x dx$

Hence, the integral reduces to

$$\begin{aligned}\int u \, du &= \frac{u^2}{2} + C \\ &= \boxed{\frac{(\ln(\cosh x))^2}{2} + C}\end{aligned}$$

(b) (10 pts) $\int \frac{e^x}{\sqrt{1-e^{2x}}} dx$

Solution:

We employ the substitution $u = e^x \Rightarrow du = e^x dx$

Hence, the integral reduces to

$$\begin{aligned}\int \frac{1}{\sqrt{1-u^2}} du &= \sin^{-1} u + C \\ &= \boxed{\sin^{-1}(e^x) + C}\end{aligned}$$

6. Consider the following function

$$y = \frac{e^x}{e^x - 2}$$

(a) (12 pts) Find the Vertical and Horizontal asymptote(s) by using the limit definitions.

Solution:

We notice that the domain of the function excludes $e^x = 2 \Rightarrow x = \ln 2$

$$\begin{aligned}\lim_{x \to \ln 2^+} y &= \lim_{x \to \ln 2^+} \frac{e^x}{e^x - 2} \\ &= \frac{e^{\ln 2}}{0+} \\ &= \infty\end{aligned}$$

and

$$\begin{aligned}\lim_{x \to \ln 2^-} y &= \lim_{x \to \ln 2^-} \frac{e^x}{e^x - 2} \\ &= \frac{e^{\ln 2}}{0-} \\ &= -\infty\end{aligned}$$

Hence there is a $\boxed{\text{vertical asymptote at } x = \ln 2}$

For horizontal asymptotes, we calculate

$$\begin{aligned}\lim_{x \to \infty} y &= \lim_{x \to \infty} \frac{e^x}{e^x - 2} \left(\frac{\infty}{\infty}\right) \\ &= \lim_{x \to \infty} \frac{e^x}{e^x} \\ &= 1\end{aligned}$$

and

$$\begin{aligned}\lim_{x \to -\infty} y &= \lim_{x \to -\infty} \frac{e^x}{e^x - 2} \\ &= \frac{0}{0-2} = 0\end{aligned}$$

Hence there are $\boxed{\text{horizontal asymptotes at } y = 0 \text{ and } y = 1}$

(b) (8 pts) Sketch the graph of the function, which should reflect the results of part(a) as well as any intercept(s).

Solution:

Clearly, there is no x intercept. The y intercept is given by

$$y(x=0) = \frac{e^0}{e^0 - 2} = -1$$

Below is a sketch of the function showing the asymptotes and the intercept.