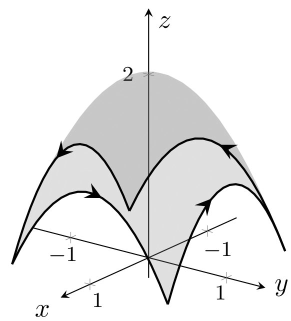
1. [2350/071726 (26 pts)] You are sitting under the tent shown in the figure below selling raffle tickets for guesses as to how many candies are in a glass jar. The tent is described by the function $z = 2 - x^2 - y^2$ over the region $-1 \le x \le 1, -1 \le y \le 1$. The wind in the area is given by the vector field $\mathbf{F}(x,y,z) = -yz\,\mathbf{i} + 2xy^2\,\mathbf{j} + 0\,\mathbf{k}$. A spider is crawling along the boundary of the tent in the direction shown and you are interested in how much work the spider does during its epic journey along the entire boundary.

   (a) (2 pts) What quadric surface describes the tent?

   (b) (2 pts) How many line integrals would be required to compute the work directly? Give a reason for your answer but do not evaluate any integrals.

   (c) (22 pts) State and use one of the Calculus III theorems to find the work without evaluating any line integrals. Zero points for using projections.



SOLUTION:

(a) circular paraboloid

(b) The boundary is not smooth, but it is piecewise smooth. Therefore, four (4) line integrals are required.

(c) Stokes' Theorem will be used. We let $\mathcal{S}$ be the tent with $\partial\mathcal{S}$ the tent's boundary along which the spider is crawling. Then

$$\text{Work} = \int_{\partial\mathcal{S}} \mathbf{F}\cdot d\mathbf{r} = \iint_{\mathcal{S}} \nabla\times\mathbf{F}\cdot\mathbf{n}\,dS$$

Begin by parameterizing $\mathcal{S}$. Since it is a function, the most straightforward parameterization is

$$\mathbf{r}(u,v) = \left\langle u, v, 2-u^2-v^2\right\rangle \implies \mathbf{r}_u = \langle 1,0,-2u\rangle, \mathbf{r}_v = \langle 0,1,-2v\rangle \implies \mathbf{r}_u\times\mathbf{r}_v = \langle 2u, 2v, 1\rangle$$

This points upward, which is the proper orientation for Stokes's theorem given the orientation of the spider's path. The parameter intervals are $-1 \le u \le 1$ and $-1 \le v \le 1$. Next,

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -yz & 2xy^2 & 0 \end{vmatrix} = 0\,\mathbf{i} - y\,\mathbf{j} + (2y^2+z)\,\mathbf{k}$$

$$\nabla\times\mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v) = \left\langle 0, -v, 2v^2+2-u^2-v^2\right\rangle\cdot\langle 2u,2v,1\rangle = -2v^2+v^2+2-u^2 = 2-u^2-v^2$$

Finally,

$$\text{Work} = \int_{-1}^{1}\int_{-1}^{1}\left(2-u^2-v^2\right)du\,dv = \int_{-1}^{1}\left[\left(2-v^2\right)u - \frac{u^3}{3}\right]\Bigg|_{-1}^{1} dv$$

$$= \int_{-1}^{1}\left[2\left(2-v^2\right)-\frac{2}{3}\right]dv = \left(\frac{10}{3}v - \frac{2v^3}{3}\right)\Bigg|_{-1}^{1} = \frac{16}{3}$$

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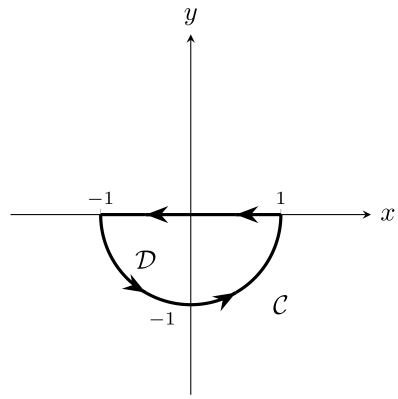
2. [2350/071726 (20 pts)] Use Green's Theorem to evaluate $\oint_{\mathcal{C}} -y^3\,\mathrm{d}x + \left(x^3 + \sqrt{y^3+1}\right)\mathrm{d}y$. $\mathcal{C}$ is the closed counterclockwise path consisting of the bottom half of the unit circle and the portion of the x-axis with $-1 \le x \le 1$.

SOLUTION:

The path and region are shown in the following figure.



$$P(x,y) = -y^3 \implies \frac{\partial P}{\partial y} = -3y^2 \quad\text{and}\quad Q(x,y) = x^3 + \sqrt{y^3+1} \implies \frac{\partial Q}{\partial x} = 3x^2$$

Thus

$$\oint_{\mathcal{C}} -y^3\,\mathrm{d}x + \left(x^3 + \sqrt{y^3+1}\right)\mathrm{d}y = \oint_{\partial\mathcal{D}} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A = \iint_{\mathcal{D}} \left(3x^2+3y^2\right)\mathrm{d}A$$

$$\overset{\text{polar}}{\underset{\text{coords}}{=}} 3\int_{\pi}^{2\pi}\int_0^1 r^3\,\mathrm{d}r\,\mathrm{d}\theta = \frac{3\pi}{4}$$

Note that this could also be considered using the flux/divergence/normal form of Green's Theorem with

$$P(x,y) = x^3 + \sqrt{y^3+1} \quad\text{and}\quad Q(x,y) = y^3$$

giving

$$\oint_{\mathcal{C}} \left(x^3 + \sqrt{y^3+1}\right)\mathrm{d}y - y^3\,\mathrm{d}x = \oint_{\partial\mathcal{D}} P\,\mathrm{d}y - Q\,\mathrm{d}x = \iint_{\mathcal{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\mathrm{d}A = \iint_{\mathcal{D}} \left(3x^2+3y^2\right)\mathrm{d}A$$

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3. [2350/071726 (23 pts)] You have a cool new "exponential" curtain covering a window in your apartment. It touches the floor ($xy-$plane) along the curve $y = e^{x/2}$ lying between $x = \ln 32$ and $x = \ln 60$. The top of the curtain reaches up to the surface $z = 3e^x$. All units are in feet. The fabric making up the curtain is sensitive to sunlight so you need to purchase some UV protection to spray on the side of the curtain facing the window. The UV protector comes in spray cans that each cover 100 square feet.

   (a) [20 pts] Use a line integral to determine how many full cans of the spray you must purchase to protect your curtain.

   (b) [3 pts] If you were provided a function $f(x,y,z)$ that gave the density (mass per unit area) of the material making up the curtain, what type of integral would you evaluate to compute the mass of the curtain?

SOLUTION:

   (a) We need to find the area of one side of the curtain, given by $\int_{\mathcal{C}} 3e^x\,\mathrm{d}s$. The base of the curtain, the function $y = e^{x/2}$, can be parameterized as

$$\mathbf{r}(t) = \left\langle t, e^{t/2}\right\rangle \quad \ln 32 \le t \le \ln 60$$

$$\implies \mathbf{r}'(t) = \left\langle 1, \tfrac{1}{2}e^{t/2}\right\rangle \implies \|\mathbf{r}'(t)\| = \sqrt{1 + \left(\tfrac{1}{2}e^{t/2}\right)^2} = \sqrt{1 + \tfrac{1}{4}e^t}$$

Then

$$\int_{\mathcal{C}} 3e^x \,\mathrm{d}s = \int_{\ln 32}^{\ln 60} 3e^t \sqrt{1+\tfrac{1}{4}e^t}\,\mathrm{d}t$$

$$u = 1+\tfrac{1}{4}e^t, \quad \mathrm{d}u = \tfrac{1}{4}e^t\,\mathrm{d}t; \quad t=\ln 32 \implies u=9; \quad t=\ln 60 \implies u=16$$

$$\int_{\mathcal{C}} 3e^x\,\mathrm{d}s = 12\int_9^{16} u^{1/2}\,\mathrm{d}u = 8u^{3/2}\Big|_9^{16} = 8(64-27) = 296 \text{ ft}^2$$

Since each can of UV protection covers 100 square feet, we need to purchase 3 cans.

(b) A scalar surface integral, $\iint_{\mathcal{S}} f(x,y,z)\,\mathrm{d}S$, where $\mathcal{S}$ is the curtain.

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4. [2350/071726 (28 pts)] The following problems are unrelated.

(a) [8 pts] Let $\mathbf{V} = (x+2y+az)\,\mathbf{i} + (bx-3y-z)\,\mathbf{j} + (4x+cy+2z)\,\mathbf{k}$. Find constants a, b and c so that $\int_{\mathcal{C}} \mathbf{V}\cdot\mathrm{d}\mathbf{r}$ is path independent.

(b) [20 pts] Find the work done by the conservative force field $\mathbf{F} = \left(2xy+z^3\right)\mathbf{i} + x^2\,\mathbf{j} + 3xz^2\,\mathbf{k}$ in moving an object along the path $\mathbf{r}(t) = \frac{4}{\pi}\left\langle \sin^{-1} t, \tan^{-1} t, \cos^{-1} t\right\rangle$, $-1 \le t \le 1$. Simplify your answer completely (*i.e.*, eliminate all trig functions)

SOLUTION:

(a) In order for the integral to be path independent, the curl of $\mathbf{V}$ must vanish on a simply connected domain. Since $\mathbf{V}$ exists on $\mathbb{R}^3$, which is simply connected, we need to force the curl to vanish throughout $\mathbb{R}^3$.

$$\nabla\times\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = (c+1)\,\mathbf{i} + (a-4)\,\mathbf{j} + (b-2)\,\mathbf{k}$$

The curl will vanish if $a=4, b=2, c=-1$.

(b) We need to compute $\int_{\mathcal{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r}$, which looks impossible given the path. However, since the vector field is conservative we can use the Fundamental Theorem for Line Integrals to get the answer. We find the potential for the vector field:

$$\frac{\partial f}{\partial x} = 2xy+z^3 \implies f(x,y,z) = \int \left(2xy+z^3\right)\,\mathrm{d}x = x^2y + xz^3 + g(y,z)$$

$$\frac{\partial f}{\partial y} = x^2 + g_y(y,z) = x^2 \implies g_y(y,z) = 0 \implies g(y,z) = h(z) \implies f(x,y,z) = x^2y + xz^3 + h(z)$$

$$\frac{\partial f}{\partial z} = 3xz^2 + h'(z) = 3xz^2 \implies h'(z) = 0 \implies h(z) = c \quad \text{which we set to 0}$$

The potential function is then $f(x,y,z) = x^2y + xz^3$. The endpoints of the path are

$$\mathbf{r}(-1) = \frac{4}{\pi}\left\langle \sin^{-1}(-1), \tan^{-1}(-1), \cos^{-1}(-1)\right\rangle = \langle -2,-1,4\rangle$$

$$\mathbf{r}(1) = \frac{4}{\pi}\left\langle \sin^{-1}(1), \tan^{-1}(1), \cos^{-1}(1)\right\rangle = \langle 2,1,0\rangle$$

We then have

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_{(-2,-1,4)}^{(2,1,0)} \nabla f\cdot\mathrm{d}\mathbf{r} = f\,(2,1,0) - f(-2,-1,4)$$

$$= 2^2(1) + 2\left(0^3\right) - \left[(-2)^2(-1) + (-2)\,(4)^3\right] = 136$$

Note that you could have integrated along any other path between the endpoints of the given curve, for example, a straight line segment, but even that one requires more algebra than finding the potential function and using the Fundamental Theorem for Line Integrals.

5. [2350/071726 (53 pts)] You need to compute the downward flux of $\mathbf{F} = \frac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k}$ through that portion of the sphere of radius 2 centered at the origin lying below the plane $z = -1$ (not a closed surface). To accomplish this feat, let $\mathcal{E}$ be the solid region between the aforementioned sphere and plane. Its closed boundary, $\partial\mathcal{E}$, consists of the union of a disk, $\mathcal{S}_{\text{disk}}$, and the surface through which we seek the flux, $\mathcal{S}_{\text{sphere}}$, that is, $\partial\mathcal{E} = \mathcal{S}_{\text{disk}} \cup \mathcal{S}_{\text{sphere}}$.

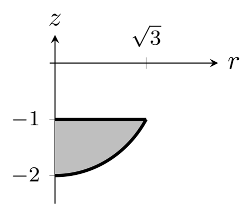
(a) [20 pts] Find the upward flux of $\mathbf{F}$ through $\mathcal{S}_{\text{disk}}$. No points for using projections.

(b) [8 pts] Show that the divergence of $\mathbf{F}$ is $\dfrac{2}{\sqrt{x^2+y^2+z^2}}$.

(c) [25 pts] Use Gauss' Divergence Theorem to find $\displaystyle\iint_{\mathcal{S}_{\text{sphere}}} \mathbf{F}\cdot \mathrm{d}\mathbf{S}$. Zero credit for calculating the flux directly.

SOLUTION:

A sketch in the rz-plane will be beneficial.



(a) $\mathcal{S}_{\text{disk}}$ has radius $\sqrt{3}$ since the intersection of the plane $z = -1$ and sphere gives $r^2 + (-1)^2 = 4 \implies r = \sqrt{3}$. This can be parameterized as $\mathbf{r}(u,v) = \left\langle \sqrt{3}u\cos v, \sqrt{3}u\sin v, -1\right\rangle, 0 \le u \le 1, 0 \le v \le 2\pi$. Then

$$\mathbf{r}_u = \left\langle \sqrt{3}\cos v, \sqrt{3}\sin v, 0\right\rangle, \quad \mathbf{r}_v = \left\langle -\sqrt{3}u\sin v, \sqrt{3}u\cos v, 0\right\rangle$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{3}\cos v & \sqrt{3}\sin v & 0 \\ -\sqrt{3}u\sin v & \sqrt{3}u\cos v & 0 \end{vmatrix} = 3u\,\mathbf{k}$$

This is properly oriented for the closed surface. We then have

$$\mathbf{F}(u,v)\cdot\mathbf{n} = \left\langle \frac{\sqrt{3}u\cos v}{\sqrt{3u^2+1}}, \frac{\sqrt{3}u\sin v}{\sqrt{3u^2+1}}, \frac{-1}{\sqrt{3u^2+1}} \right\rangle \cdot \langle 0,0,3u\rangle = \frac{-3u}{\sqrt{3u^2+1}}$$

$$\text{Flux} = \iint_{\mathcal{S}_{\text{disk}}} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \int_0^{2\pi}\int_0^1 \frac{-3u}{\sqrt{3u^2+1}}\,\mathrm{d}u\,\mathrm{d}v \overset{w=3u^2+1}{=} -\int_0^{2\pi}\int_1^4 \frac{1}{2}w^{-1/2}\,\mathrm{d}w\,\mathrm{d}v$$

$$= -\int_0^{2\pi} w^{1/2}\Big|_1^4\,\mathrm{d}v = -\int_0^{2\pi}\mathrm{d}v = -2\pi$$

Alternatively, the parameterization $\mathbf{r}(u,v) = \langle u, v, -1\rangle$ with $u^2+v^2 \le 3$ can be used.

(b)

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{\sqrt{x^2+y^2+z^2} - x\left(\frac{2x}{2\sqrt{x^2+y^2+z^2}}\right)}{x^2+y^2+z^2} + \frac{\sqrt{x^2+y^2+z^2} - y\left(\frac{2y}{2\sqrt{x^2+y^2+z^2}}\right)}{x^2+y^2+z^2} + \frac{\sqrt{x^2+y^2+z^2} - z\left(\frac{2z}{2\sqrt{x^2+y^2+z^2}}\right)}{x^2+y^2+z^2}$$

$$= \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{2x^2+2y^2+2z^2}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}} = \frac{2}{\sqrt{x^2+y^2+z^2}}$$

(c) Gauss' Divergence Theorem is $\iiint_{\mathcal{E}} \nabla \cdot \mathbf{F}\, \mathrm{d}V = \iint_{\partial\mathcal{E}} \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}S = \iint_{\mathcal{S}_{\text{sphere}}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} + \iint_{\mathcal{S}_{\text{disk}}} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ so that

$$\iint_{\mathcal{S}_{\text{sphere}}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_{\mathcal{E}} \nabla \cdot \mathbf{F}\, \mathrm{d}V - \iint_{\mathcal{S}_{\text{disk}}} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

Now, transforming to spherical coordinates we have

$$\begin{aligned}
\iiint_{\mathcal{E}} \nabla \cdot \mathbf{F}\, \mathrm{d}V &= \iiint_{\mathcal{E}} \frac{2}{\sqrt{x^2+y^2+z^2}}\, \mathrm{d}V = \int_0^{2\pi} \int_{2\pi/3}^{\pi} \int_{-\sec\phi}^{2} \frac{2}{\rho} \rho^2 \sin\phi \,\mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\theta \\
&= \left(\int_0^{2\pi} \mathrm{d}\theta \right) \int_{2\pi/3}^{\pi} \int_{-\sec\phi}^{2} 2\rho \sin\phi \,\mathrm{d}\rho\, \mathrm{d}\phi = 2\pi \int_{2\pi/3}^{\pi} \rho^2 \Big|_{-\sec\phi}^{2} \sin\phi\, \mathrm{d}\phi \\
&= 2\pi \int_{2\pi/3}^{\pi} \left(4\sin\phi - \sec^2\phi \sin\phi\right) \mathrm{d}\phi = 2\pi \left(4\cos\phi \Big|_{\pi}^{2\pi/3} - \int_{2\pi/3}^{\pi} \sec\phi \tan\phi\, \mathrm{d}\phi \right) \\
&= 2\pi \left(2 - \sec\phi \Big|_{2\pi/3}^{\pi} \right) = 2\pi
\end{aligned}$$

Finally then,

$$\iint_{\mathcal{S}_{\text{sphere}}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = 2\pi - (-2\pi) = 4\pi$$

Alternatively, cylindrical coordinates give

$$\int_0^{2\pi} \int_0^{\sqrt{3}} \int_{-\sqrt{4-r^2}}^{-1} \frac{2}{\sqrt{r^2+z^2}}\, r\, \mathrm{d}z\, \mathrm{d}r\, \mathrm{d}\theta \qquad \text{difficult integral}$$

or

$$\begin{aligned}
&\int_0^{2\pi} \int_{-2}^{-1} \int_0^{\sqrt{4-z^2}} \frac{2}{\sqrt{r^2+z^2}}\, r\, \mathrm{d}r\, \mathrm{d}z\, \mathrm{d}\theta \overset{u=r^2+z^2}{=} \int_0^{2\pi} \int_{-2}^{-1} \int_{z^2}^{4} u^{-1/2}\, \mathrm{d}u\, \mathrm{d}z\, \mathrm{d}\theta = 2\pi \int_{-2}^{-1} 2\sqrt{u}\, \Big|_{z^2}^{4} \mathrm{d}z \\
&= 4\pi \int_{-2}^{-1} (2 - |z|)\, \mathrm{d}z = 4\pi \int_{-2}^{-1} (2+z)\, \mathrm{d}z = 4\pi \left(2z + \tfrac{z^2}{2} \right) \Big|_{-2}^{-1} = 4\pi \left[-2 + \tfrac{1}{2} - (-4+2) \right] = 2\pi
\end{aligned}$$

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