1. [2360/071726 (24 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a two-column table (letter - answer) completely separate from any work you do to arrive at the answer.

   (a) Picard's theorem guarantees that the initial value problem $y' = \sqrt[3]{ty},\ y(1) = 0$ has at least two solutions.

   (b) Euler's method will produce the exact solution to any initial value problem of the form $y' = c,\ y(t_0) = y_0$, where c, t_0, y_0 are real numbers, regardless of the step size.

   (c) If $f(t) = \delta(t) + \delta(t-1) + \delta(t-2) + \cdots$, then $\mathscr{L}\{f(t)\} = \displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{e^s}\right)^n$

   (d) $\mathbb{R}^3 \neq \text{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$

   (e) The system $\begin{array}{l} x' = y \\ y' = 1 - x^2 \end{array}$ possesses three equilibrium solutions.

   (f) The differential equation $y' - (t+1)y = -2(t+1)$ has no equilibrium solutions.

   (g) The set, $\mathbb{W}$, of $n \times n$ skew-symmetric matrices, those for which $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$, is a subspace of the vector space $\mathbb{M}_{nn}$.

   (h) $y(t)$ is bounded over all time t if $y(t)$ solves $y'' + y = \cos t$.

   SOLUTION:

   (a) **FALSE** Although $f(t, y) = \sqrt[3]{ty}$ is continuous for all t, y (implying the existence of at least one solution to the IVP), $f_y(t, y) = \frac{t^{1/3}}{3y^{2/3}}$ is not defined at $(1, 0)$. Therefore, it is not continuous in a rectangle containing $(1, 0)$. Picard's theorem then guarantees nothing about the number of solutions.

   (b) **TRUE** The exact solution from separation of variables is $y(t) = y_0 + c(t - t_0)$. Euler's method is $y_{n+1} = y_n + hc$, giving

$$\begin{aligned} y_1 &= y_0 + hc \\ y_2 &= y_1 + hc = (y_0 + hc) + hc = y_0 + 2hc \\ y_3 &= y_2 + hc = (y_0 + 2hc) + hc = y_0 + 3hc \\ &\vdots \\ y_n &= y_0 + nhc = y_0 + c(t_n - t_0) \end{aligned}$$

   since $t_n = t_0 + nh$. This is simply the exact solution evaluated at t_n, that is, $y(t_n) = y_0 + c(t_n - t_0)$.

   (c) **TRUE** $\mathscr{L}\{f(t)\} = e^{-s} + e^{-2s} + e^{-3s} + \cdots = \displaystyle\sum_{n=0}^{\infty} e^{-ns} = \sum_{n=0}^{\infty}\left(e^{-s}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{e^s}\right)^n$

   (d) **TRUE** Although there are three vectors in a vector space of dimension 3, the vectors are not linearly independent and thus cannot be a basis and therefore cannot span $\mathbb{R}^3$.

   (e) **FALSE** The v nullcline is $y = 0$. The h nullclines are $x = \pm 1$. The equilibrium solutions are $(1, 0)$ and $(-1, 0)$.

   (f) **FALSE** The equation can be rewritten as $y' = (y-2)(t+1)$ showing that $y = 2$ is an equilibrium solution.

   (g) **TRUE** Let $\mathbf{A}, \mathbf{B} \in \mathbb{W}$ and $a, b \in \mathbb{R}$. Then

$$(a\mathbf{A} + b\mathbf{B})^{\mathrm{T}} = (a\mathbf{A})^{\mathrm{T}} + (b\mathbf{B})^{\mathrm{T}} = a\mathbf{A}^{\mathrm{T}} + b\mathbf{B}^{\mathrm{T}} = a(-\mathbf{A}) + b(-\mathbf{B}) = -(a\mathbf{A} + b\mathbf{B}) \implies a\mathbf{A} + b\mathbf{B} \in \mathbb{W}$$

   (h) **FALSE** This differential equation describes an oscillator in resonance, the solutions of which are unbounded with particular solutions of the form $At \sin t + Bt \cos t$.

2. [2360/071726 (24 pts)] Consider the linear system $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A}$ is a 2×2 matrix with $\operatorname{Tr} \mathbf{A} = 2p$, $|\mathbf{A}| = p^2 - p - 2$, and p is a real number. Recall that fixed points and equilibrium solutions are essentially the same things. No work need be shown and no partial credit is available.

   (a) For what values, if any, of p does the system have a unique equilibrium solution at $(0, 0)$?

(b) For what values, if any, of p will the trajectories in the phase plane be closed loops?

(c) For what values, if any, of p will the equilibrium solution be unstable?

(d) For what values, if any, of p will the system have fixed points other than $(0, 0)$?

(e) For what values, if any, of p will the equilibrium solution be a degenerate or star node?

(f) For what values, if any, of p will $\lim_{t\to\infty} \vec{\mathbf{x}}(t) = \vec{\mathbf{0}}$?

(g) For what values, if any, of p will the equilibrium solution be a saddle?

(h) For what values, if any, of p will the equilibrium solution be a repelling spiral?

SOLUTION:

Note that $\Delta = (\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| = (2p)^2 - 4(p^2 - p - 2) = 4(p + 2)$ and $|\mathbf{A}| = (p - 2)(p + 1)$.

(a) need $|\mathbf{A}| \neq 0 \implies \boxed{p \neq -1, 2}$

(b) need $\text{Tr}\,\mathbf{A} = 0 \implies p = 0$ and $|\mathbf{A}| > 0 \implies (-\infty, -1) \cup (2, \infty) \implies \boxed{\text{NONE}}$

(c) need $\text{Tr}\,\mathbf{A} > 0 \implies p > 0$ or $|\mathbf{A}| < 0 \implies -1 < p < 2 \implies \boxed{p > -1}$

(d) need $|\mathbf{A}| = 0 \implies \boxed{p = -1, 2}$

(e) need $|\mathbf{A}| > 0 \implies (-\infty, -1) \cup (2, \infty)$ and $\Delta = 0 \implies p = -2 \implies \boxed{p = -2}$

(f) need $\text{Tr}\,\mathbf{A} < 0 \implies p < 0$ and $|\mathbf{A}| > 0 \implies (-\infty, -1) \cup (2, \infty) \implies \boxed{p < -1}$

(g) need $|\mathbf{A}| < 0 \implies \boxed{-1 < p < 2}$

(h) need $\text{Tr}\,\mathbf{A} > 0 \implies p > 0$ and $|\mathbf{A}| > 0 \implies (-\infty, -1) \cup (2, \infty)$ and $\Delta < 0 \implies p < -2 \implies \boxed{\text{None}}$

3. [2360/071726 (45 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix} -3 & 0 & 1 \\ -2 & -2 & 2 \\ -9 & 0 & 3 \end{bmatrix}$.

(a) (5 pts) Show that $\mathbf{A}$ is singular.

(b) (5 pts) Does the system of equations $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ have at least two solutions? Justify your answer.

(c) (5 pts) If $\vec{\mathbf{b}} = [0\ \ 0\ \ 1]^{\mathrm{T}}$, is $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ consistent? Justify your answer.

(d) (5 pts) Show that $\mathbf{A}$ has two eigenvalues, neither of which is positive. Determine the algebraic multiplicity of each eigenvalue. Verify your calculations since the remaining part depends on your answer here.

(e) (25 pts) Solve the initial value problem $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = [1\ \ 5\ \ 5]^{\mathrm{T}}$. Write your answer as a single vector.

SOLUTION:

(a)

$$|\mathbf{A}| = -2(-1)^{2+2} \begin{vmatrix} -3 & 1 \\ -9 & 3 \end{vmatrix} = -2(-9 - (-9) = 0$$

There are other ways to show this.

(b) Yes. Since $\mathbf{A}$ is singular, equivalently the homogeneous system has infinitely many (nontrivial) solutions, hence more than 1 or at least 2 (see Linear Algebra Theorem in Canvas)

(c) No. After one elementary row operation we get

$$\left[\begin{array}{ccc|c} -3 & 0 & 1 & 0 \\ -2 & -2 & 2 & 0 \\ -9 & 0 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} -3 & 0 & 1 & 0 \\ -2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The last row shows that the system has no solution and thus is inconsistent.

(d)

$$\begin{vmatrix} -3-\lambda & 0 & 1 \\ -2 & -2-\lambda & 2 \\ -9 & 0 & 3-\lambda \end{vmatrix} = (-2-\lambda)(-1)^{2+2}\begin{vmatrix} -3-\lambda & 1 \\ -9 & 3-\lambda \end{vmatrix}$$

$$= (-2-\lambda)\left[(-3-\lambda)(3-\lambda)+9\right] = (-2-\lambda)(\lambda^2) = -\lambda^2(\lambda+2)$$

The eigenvalues are $\lambda = 0$ with algebraic multiplicity 2 and $\lambda = -2$ with algebraic multiplicity of 1.

(e) We need to find the eigenvectors associated with the eigenvalues previously found. For $\lambda = 0$, solve $\mathbf{A}\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

$$\left[\begin{array}{rrr|r} -3 & 0 & 1 & 0 \\ -2 & -2 & 2 & 0 \\ -9 & 0 & 3 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} -3 & 0 & 1 & 0 \\ -2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -\frac{1}{3} & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{array}{r} v_1 = \frac{1}{3}v_3 \\ v_2 = \frac{2}{3}v_3 \\ v_3 = t \end{array}$$

With $t = 3$, the eigenvector is $\vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Since there is only one eigenvector, we need to find the generalized eigenvector by solving $\mathbf{A}\vec{\mathbf{u}} = \vec{\mathbf{v}}$.

$$\left[\begin{array}{rrr|r} -3 & 0 & 1 & 1 \\ -2 & -2 & 2 & 2 \\ -9 & 0 & 3 & 3 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -\frac{1}{3} & -\frac{1}{3} \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{array}{l} u_1 = -\frac{1}{3}+\frac{1}{3}u_3 \\ u_2 = -\frac{2}{3}+\frac{2}{3}u_3 \\ u_3 = t \end{array}$$

With $t = 1$, the generalized eigenvector is $\vec{\mathbf{u}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. For $\lambda = -2$, we solve $(\mathbf{A}+2\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

$$\left[\begin{array}{rrr|r} -1 & 0 & 1 & 0 \\ -2 & 0 & 2 & 0 \\ -9 & 0 & 5 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{array}{r} v_1 = 0 \\ v_2 = t \\ v_3 = 0 \end{array}$$

With $t = 1$, the eigenvector is $\vec{\mathbf{v}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. The general solution is then

$$\vec{\mathbf{x}}(t) = c_1 e^{-2t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_3\left(t\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

Applying the initial condition yields

$$c_1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix} \to \begin{array}{r} c_2 = 1 \\ c_1 + 2c_2 = 5 \\ 3c_2 + c_3 = 5 \end{array} \implies \begin{array}{r} c_1 = 3 \\ c_2 = 1 \\ c_3 = 2 \end{array}$$

giving the following as the solution to the initial value problem

$$\vec{\mathbf{x}} = 3e^{-2t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\left(t\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2t+1 \\ 3e^{-2t}+4t+2 \\ 6t+5 \end{bmatrix}$$

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4. [2360/071726 (27 pts)] Consider the initial value problem $y'' + 2y' = 4,\ y(0) = 3, y'(0) = 4$.

(a) (25 pts) Use Laplace transforms to solve the IVP.

(b) (1 pt) What would be the guess for the particular solution if using the Method of Undetermined Coefficient to solve the IVP? Just write down the guess.

(c) (1 pt) Can you name another technique that we studied in this class that could be used to solve the IVP? If there is, name it. If not, just say no.

SOLUTION:

(a)

$$s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] = \frac{4}{s}$$

$$\left(s^2+2s\right)Y(s) = \frac{4}{s} + sy(0) + y'(0) + 2y(0) = \frac{4}{s} + 3s + 4 + 6 = \frac{4}{s} + 3s + 10 = \frac{3s^2+10s+4}{s}$$

$$Y(s) = \frac{3s^2+10s+4}{s^2(s+2)}$$

$$\frac{3s^2+10s+4}{s^2(s+2)} = \frac{As+B}{s^2} + \frac{C}{s+2}$$

$$3s^2+10s+4 = (As+B)(s+2) + Cs^2$$

$$s = 0 : 4 = 2B \implies B = 2$$

$$s = -2 : -4 = 4C \implies C = -1$$

$$s = 1 : 17 = (A+2)(3) - 1 \implies 18 = 3(A+2) \implies 6 = A+2 \implies A = 4$$

$$Y(s) = \frac{3s^2+10s+4}{s^2(s+2)} = \frac{4s+2}{s^2} + \frac{-1}{s+2} = \frac{4}{s} + \frac{2}{s^2} - \frac{1}{s+2}$$

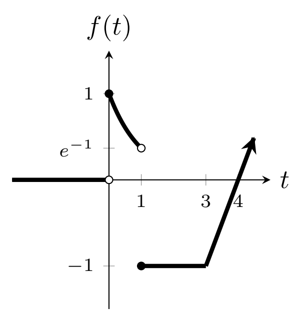
$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{4}{s} + \frac{2}{s^2} - \frac{1}{s+2}\right\} = 4 + 2t - e^{-2t}$$

(b) The characteristic equation is $r^2+2r = r(r+2) = 0 \implies r = 0, -2$. The guess for the particular solution is $y_p = At$.

(c) Yes, Variation of Parameters.

5. [2360/071726 (30 pts)] The following parts are not related.

(a) (10 pts) The graph of $f(t)$ is shown. Write $f(t)$ using step functions, combining like terms as much as possible.



(b) (10 pts) Evaluate $\mathscr{L}^{-1}\left\{\dfrac{se^{-2\pi s}}{s^2+4s+5}\right\}$. Recall the periodicity of sine and cosine and simplify your answer.

(c) (10 pts) Find the Laplace transform of $f(t) = te^t\text{step}(t-1)$, simplifying your final answer and writing it as a single fraction.

SOLUTION:

(a) The function written piecewise is

$$f(t) = \begin{cases} 0, & t < 0 \\ e^{-t}, & 0 \le t < 1 \\ -1, & 1 \le t < 3 \\ t-4, & t \ge 3 \end{cases}$$

Using step functions this is

$$f(t) = e^{-t}\text{step}(t) - e^{-t}\text{step}(t-1) - \text{step}(t-1) + \text{step}(t-3) + (t-4)\text{step}(t-3)$$

or, combining like terms,

$$f(t) = e^{-t}\left[\text{step}(t) - \text{step}(t-1)\right] - \text{step}(t-1) + (t-3)\text{step}(t-3)$$

or, combining like terms differently,

$$f(t) = e^{-t}\text{step}(t) - \left(e^{-t}+1\right)\text{step}(t-1) + (t-3)\text{step}(t-3)$$

(b) Begin by noting that

$$\frac{s}{s^2+4s+5} = \frac{s}{s^2+4s+4-4+5} = \frac{s}{(s+2)^2+1} = \frac{s+2-2}{(s+2)^2+1} = \frac{s+2}{(s+2)^2+1} - 2\frac{1}{(s+2)^2+1}$$

Then

$$\mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1} - 2\frac{1}{(s+2)^2+1}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\bigg|_{s\to s+2} - 2\frac{1}{s^2+1}\bigg|_{s\to s+2}\right\}$$
$$= e^{-2t}\left(\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - 2\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\right) = e^{-2t}\left(\cos t - 2\sin t\right)$$

Now impose the delay and note that the period of sine and cosine is 2π.

$$\mathscr{L}^{-1}\left\{\frac{se^{-2\pi s}}{s^2+4s+5}\right\} = \text{step}(t-2\pi)e^{-2(t-2\pi)}\left[\cos(t-2\pi) - 2\sin(t-2\pi)\right] = e^{-2(t-2\pi)}\left(\cos t - 2\sin t\right)\text{step}(t-2\pi)$$

(c)

$$\mathscr{L}\left\{te^t\text{step}(t-1)\right\} = e^{-s}\mathscr{L}\left\{(t+1)e^{t+1}\right\} = e^{1-s}\mathscr{L}\left\{te^t + e^t\right\}$$
$$= e^{1-s}\left(\mathscr{L}\left\{t\right\}\Big|_{s\to s-1} + \frac{1}{s-1}\right) = e^{1-s}\left(\frac{1}{s^2}\bigg|_{s\to s-1} + \frac{1}{s-1}\right)$$
$$= e^{1-s}\left[\frac{1}{(s-1)^2} + \frac{1}{s-1}\right] = e^{1-s}\left[\frac{1+s-1}{(s-1)^2}\right] = \frac{se^{1-s}}{(s-1)^2}$$

Note that we also could have done the following:

$$\mathscr{L}\left\{te^t\right\} = (-1)^1\frac{\mathrm{d}}{\mathrm{d}s}\mathscr{L}\left\{e^t\right\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s-1}\right) = \frac{1}{(s-1)^2}$$

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