- This exam is worth 150 points and has 5 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5”× 11” crib sheet with writing on both sides.

0. At the top of the page containing your solution to problem 1, write the following statement and sign your name to it: “I will abide by the CU Boulder Honor Code on this exam.” **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/071726 (24 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a two-column table (letter - answer) completely separate from any work you do to arrive at the answer.

   (a) Picard’s theorem guarantees that the initial value problem $y' = \sqrt[3]{ty},\ y(1) = 0$ has at least two solutions.

   (b) Euler’s method will produce the exact solution to any initial value problem of the form $y' = c,\ y(t_0) = y_0$, where c, t_0, y_0 are real numbers, regardless of the step size.

   (c) If $f(t) = \delta(t) + \delta(t-1) + \delta(t-2) + \cdots$, then $\mathscr{L}\{f(t)\} = \sum_{n=0}^{\infty} \left(\frac{1}{e^s}\right)^n$

   (d) $\mathbb{R}^3 \neq \text{span}\left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix} \right\}$

   (e) The system $\begin{aligned} x' &= y \\ y' &= 1 - x^2 \end{aligned}$ possesses three equilibrium solutions.

   (f) The differential equation $y' - (t+1)y = -2(t+1)$ has no equilibrium solutions.

   (g) The set, $\mathbb{W}$, of $n \times n$ skew-symmetric matrices, those for which $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$, is a subspace of the vector space $\mathbb{M}_{nn}$.

   (h) $y(t)$ is bounded over all time t if $y(t)$ solves $y'' + y = \cos t$.

2. [2360/071726 (24 pts)] Consider the linear system $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A}$ is a 2×2 matrix with $\operatorname{Tr}\mathbf{A} = 2p, |\mathbf{A}| = p^2 - p - 2$, and p is a real number. Recall that fixed points and equilibrium solutions are essentially the same things. No work need be shown and no partial credit is available.

   (a) For what values, if any, of p does the system have a unique equilibrium solution at $(0,0)$?

   (b) For what values, if any, of p will the trajectories in the phase plane be closed loops?

   (c) For what values, if any, of p will the equilibrium solution be unstable?

   (d) For what values, if any, of p will the system have fixed points other than $(0,0)$?

   (e) For what values, if any, of p will the equilibrium solution be a degenerate or star node?

   (f) For what values, if any, of p will $\lim_{t\to\infty} \vec{\mathbf{x}}(t) = \vec{\mathbf{0}}$?

   (g) For what values, if any, of p will the equilibrium solution be a saddle?

   (h) For what values, if any, of p will the equilibrium solution be a repelling spiral?

MORE PROBLEMS AND LAPLACE TRANSFORM TABLE BELOW/ON REVERSE

3. [2360/071726 (45 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix} -3 & 0 & 1 \\ -2 & -2 & 2 \\ -9 & 0 & 3 \end{bmatrix}$.

(a) (5 pts) Show that $\mathbf{A}$ is singular.

(b) (5 pts) Does the system of equations $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ have at least two solutions? Justify your answer.

(c) (5 pts) If $\vec{\mathbf{b}} = [0\ \ 0\ \ 1]^{\mathrm{T}}$, is $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ consistent? Justify your answer.

(d) (5 pts) Show that $\mathbf{A}$ has two eigenvalues, neither of which is positive. Determine the algebraic multiplicity of each eigenvalue. Verify your calculations since the remaining part depends on your answer here.

(e) (25 pts) Solve the initial value problem $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = [1\ \ 5\ \ 5]^{\mathrm{T}}$. Write your answer as a single vector.

4. [2360/071726 (27 pts)] Consider the initial value problem $y'' + 2y' = 4,\ y(0) = 3, y'(0) = 4$.

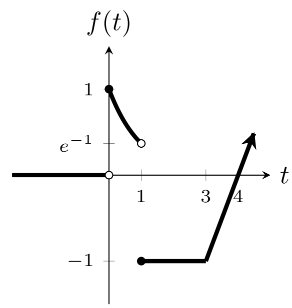
(a) (25 pts) Use Laplace transforms to solve the IVP.

(b) (1 pt) What would the guess for the particular solution be if using the Method of Undetermined Coefficient to solve the IVP? Just write down the guess.

(c) (1 pt) Can you name another technique that we studied in this class that could be used to solve the IVP? If there is, name it. If not, just say no.

5. [2360/071726 (30 pts)] The following parts are not related.

(a) (10 pts) The graph of $f(t)$ is shown. Write $f(t)$ using step functions, combining like terms as much as possible.



(b) (10 pts) Evaluate $\mathscr{L}^{-1}\left\{\dfrac{se^{-2\pi s}}{s^2+4s+5}\right\}$. Recall the periodicity of sine and cosine and simplify your answer.

(c) (10 pts) Find the Laplace transform of $f(t) = te^t\text{step}(t-1)$, simplifying your final answer and writing it as a single fraction.

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{\cosh bt\} = \frac{s}{s^2-b^2} \qquad \mathscr{L}\{\sinh bt\} = \frac{b}{s^2-b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{tf'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\text{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\text{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$