Write your name below. This exam is worth 100 points and has 6 problems. You must show your work to receive full credit on the problems. You are allowed to use one side of an 8.5x11 page of notes. You cannot collaborate on the exam or seek outside help, nor can you use the recorded lectures, a calculator, any computational software, or material you find online.

Name:

1. (24 points, 8 each) In each of the following problems, either show that the statement is always true or provide a counterexample that shows it is false.

   (a) The $p = 1$ norm on $\mathbb{R}^2$, $||\cdot||_1$, does not have an associated inner product.

   (b) Let $K = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$, which factors into $\begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$. The vector product defined as $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T K \overline{\mathbf{y}}$ is an inner product on $\mathbb{C}^2$.

   (c) For every matrix A, $||A||_\infty \leq ||A||_1$.

Solution: (a) **True.** If this norm has an inner product, then it must satisfy the parallelogram law for all vectors in $\mathbb{R}^2$:

$$||\mathbf{x} + \mathbf{y}||_1^2 + ||\mathbf{x} - \mathbf{y}||_1^2 = 2(||\mathbf{x}||^2 + ||\mathbf{y}||^2)$$

If we use the two vectors

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{y} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Then we have

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \text{ and } \mathbf{x} - \mathbf{y} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

and we find that

$$\begin{aligned} ||\mathbf{x} + \mathbf{y}||_1^2 + ||\mathbf{x} - \mathbf{y}||_1^2 &= (2+1) + (0+3) = 6 \\ 2(||\mathbf{x}||^2 + ||\mathbf{y}||^2) &= 2((1+1) + (1+2)) = 10 \end{aligned}$$

Since the parallelogram law does not hold for these vectors, the 1-norm does not have an associated inner product.

(b) **True.** This vector product satisfies all three axioms. Writing $K = LD\overline{L^T}$, we see that it has conjugate symmetry:

$$\begin{aligned} \overline{\langle \mathbf{y}, \mathbf{x}\rangle} &= \overline{\mathbf{y}^T LD\overline{L^T}\overline{\mathbf{x}}} \\ &= \overline{\mathbf{y}^T L} DL^T \mathbf{x} \\ &= (\overline{\mathbf{y}^T L} DL^T \mathbf{x})^T && \text{(scalars are equal to their transposes)} \\ &= \mathbf{x}^T LD\overline{L^T}\mathbf{y} \\ &= \langle \mathbf{x}, \mathbf{y}\rangle \end{aligned}$$

It is positive, since all of the values of :

$$\begin{aligned} \langle \mathbf{x}, \mathbf{x}\rangle &= \mathbf{x}^T LD\overline{L^T}\overline{\mathbf{x}} \\ &= ||L^T\mathbf{x}||^2 > 0 \text{ for } \mathbf{x} \neq \mathbf{0} \end{aligned}$$

And the product is sesquilinear:

$$\begin{aligned} \langle a\mathbf{x} + b\mathbf{z}, \mathbf{y}\rangle &= (a\mathbf{x} + b\mathbf{z})K\overline{\mathbf{y}} = a\mathbf{x}K\overline{\mathbf{y}} + b\mathbf{z}K\overline{\mathbf{y}} = a\langle \mathbf{x}, \mathbf{y}\rangle + b\langle \mathbf{z}, \mathbf{y}\rangle \\ \langle \mathbf{x}, a\mathbf{y} + b\mathbf{z}\rangle &= \mathbf{x}K\overline{(a\mathbf{y} + b\mathbf{z})} = \overline{a}\mathbf{x}K\overline{\mathbf{y}} + \overline{b}\mathbf{z}K\overline{\mathbf{y}} = \overline{a}\langle \mathbf{x}, \mathbf{y}\rangle + \overline{b}\langle \mathbf{z}, \mathbf{y}\rangle \end{aligned}$$

(c) **False.** A counter example is $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, which has

$$||A||_\infty = 2$$
$$||A||_1 = 1$$

2. (14 points) You have calculated the REF of matrix A and found that

$$A = LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 \\ 3 & -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 7 & 0 & 1 & 1 \\ 0 & 0 & 1 & 8 & 6 & 4 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

(a) (5 points) Find a basis for $\ker A$.

(b) (5 points) Is the vector $\begin{pmatrix} 2 & 1 & 8 & 8 & 1 & -2 \end{pmatrix}^T$ in the coimage of A?

(c) (4 points) What are the dimensions of all 4 fundamental subspaces of A?

Solution: (a) We have two free variables in the REF, x_2 and x_4, so there will be two basis vectors for the kernel. We solve the homogeneous equation $U\mathbf{x} = \mathbf{0}$. First setting $x_2 = 1$ and $x_4 = 0$, the rows give :

$$\begin{aligned} 2x_1 + 1 \cdot 1 + 7x_3 + 0 + x_5 + x_6 &= 0 \\ x_3 + 0 + 6x_5 + 4x_6 &= 0 \\ 3x_5 &= 0 \\ 7x_6 &= 0 \end{aligned}$$

The last two equations tell us that $x_6 = 0$ and $x_5 = 0$. Substituting these values into the second gives $x_3 = 0$. We are left with

$$\begin{aligned} 2x_1 + 1 &= 0 \\ x_1 &= -\frac{1}{2} \end{aligned}$$

So the first basis vector of the kernel is $z_1 = \begin{pmatrix} -1/2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

For the second basis vector we set $x_2 = 0$ and $x_4 = 1$. Now the equations we get are

$$\begin{aligned} 2x_1 + 1 \cdot 0 + 7x_3 + 0 \cdot 1 + x_5 + x_6 &= 0 \\ x_3 + 8 \cdot 1 + 6x_5 + 4x_6 &= 0 \\ 3x_5 &= 0 \\ 7x_6 &= 0 \end{aligned}$$

Again, the last two equations tell us that $x_6 = 0$ and $x_5 = 0$. Now substituting these values into the second gives

$$\begin{aligned} x_3 + 8 &= 0 \\ x_3 &= -8 \end{aligned}$$

Now substituting into the first equation gives

$$\begin{aligned} 2x_1 + 7(-8) &= 0 \\ x_1 &= 28 \end{aligned}$$

So the second basis vector of the kernel is $z_2 = \begin{pmatrix} 28 \\ 0 \\ -8 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

(b) The basis vectors for the coimage of A are the transposed rows of U. To see if this vector is in the coimage, we check if it is in the span of the basis vectors:

$$\left(\begin{array}{cccc|c} 2 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 7 & 1 & 0 & 0 & 8 \\ 0 & 8 & 0 & 0 & 8 \\ 1 & 6 & 3 & 0 & 1 \\ 1 & 4 & 0 & 7 & -2 \end{array}\right) \xrightarrow[R_1 \to \frac{1}{2}R_1]{} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 7 & 1 & 0 & 0 & 8 \\ 0 & 8 & 0 & 0 & 8 \\ 1 & 6 & 3 & 0 & 1 \\ 1 & 4 & 0 & 7 & -2 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 8 & 0 & 0 & 8 \\ 0 & 6 & 3 & 0 & 0 \\ 0 & 4 & 0 & 7 & -3 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & -6 \\ 0 & 0 & 0 & 7 & -7 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & -6 \\ 0 & 0 & 0 & 7 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The system is compatible and so has a solution. Since the solution exists, the vector is in the span, and so is in the coimage of A

One could also show that the vector is orthogonal to both basis vectors of the kernel.

(c) The matrix is rank 4, as it has 4 pivots, so dim img $A =$ dim coimg $A = 4$ by the fundamental theorem. The kernel has 2 basis vectors, so dim $\ker A = 2$. Finally, since the dimension of the image plus the dimension of the cokernel must equal the number of rows of the matrix, we have dim coker $A = 0$.

3. (18 points) Let $K = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{pmatrix}$.

   (a) (6 points) What is the quadratic form $q(\mathbf{x}) = \mathbf{x}^T K \mathbf{x}$?

   (b) (10 points) Find the LDL^T factorization of K.

   (c) (2 points) Is the vector product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T K \mathbf{y}$ an inner product on $\mathbb{R}^3$?

Solution: (a) We can read off the quadratic form from the entries in K:

$$q(\mathbf{x}) = x^2 + 2xy + 2xz + 3y^2 + 6yz + 4z^2$$

(b) First factor K into LU by row reduction:

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Now let $U = DV$. D is the diagonal matrix with the pivots for entries. Dividing each row of U by it's pivot gives us V. The factorization of K is therefore:

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) Yes. Since K only has positive pivots, it is positive definite, $K > 0$. Therefore the vector product is a valid inner product.

4. (14 points) The following questions refer to the set of polynomials

$$\left\{x^2+1,\ x,\ x^2-1\right\}$$

(a) (4 points) Show that these polynomials are a basis for $\mathcal{P}^{(2)}$, the vector space of polynomials with degree ≤ 2.

(b) (5 points) Using the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$, are these an orthogonal basis? (Hint: You may use symmetry to simplify your calculations.)

(c) (5 points) If the basis is not orthogonal, use the Gram-Schmidt procedure to find an orthogonal basis using the inner product from part (b).

Solution: (a) We write an arbitrary member of $\mathcal{P}^{(2)}$ as a linear combination of these polynomials and determine if there is always a solution to the resulting system:

$$\begin{aligned} c_1(x^2+1) + c_2(x) + c_3(x^2-1) &= ax^2+bx+d \\ (c_1+c_3)x^2 + (c_2)x + (c_1-c_3) &= ax^2+bx+d \end{aligned}$$

Our system becomes

$$\begin{aligned} c_1 + c_3 &= a \\ c_2 &= b \\ c_1 - c_3 &= d \end{aligned}$$

The coefficient matrix is non-singular:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

so the polynomials are linearly independent. As there are three of them, and $\mathcal{P}^{(2)}$ has dimension $2+1=3$, these polynomials are a basis for $\mathcal{P}^{(2)}$

(b) As we are integrating from -1 to 1, any product of the functions where the integrand is odd will have an inner product of 0. We see that

$$\begin{aligned} \langle x^2+1, x\rangle &= \int_{-1}^{1} x^3 + x\ dx = 0 \\ \langle x^2-1, x\rangle &= \int_{-1}^{1} x^3 - x\ dx = 0 \end{aligned}$$

but

$$\langle x^2+1, x^2-1\rangle = \int_{-1}^{1} x^4 - 1dx = \left(\frac{1}{5}x^5 - x\right)\Big|_{-1}^{1} = -\frac{8}{5} \neq 0$$

so they are not an orthogonal basis.

(c) We saw in part b that the first and second vector are orthogonal, so we may take them as the first two vectors in out basis. We find the third vector by subtracting the component along x^2+1 from the third. (We already know the component along x is zero from part b.)

$$\begin{aligned}\langle x^2+1, x^2+1\rangle &= \int_{-1}^{1} x^4+2x^2+1dx = \left(\frac{1}{5}x^5+\frac{2}{3}x^3+x\right)\Big|_{-1}^{1}\\ &= \frac{56}{15}\end{aligned}$$

so our third vector is

$$\begin{aligned}x^2-1-\frac{\left(-\frac{8}{5}\right)}{\left(\frac{56}{15}\right)}(x^2+1) &= x^2-1+\left(\frac{8}{5}\right)\left(\frac{15}{56}\right)(x^2+1)\\ &= x^2-1+\frac{3}{7}(x^2+1)\\ &= \frac{1}{7}(10x^2-4)\end{aligned}$$

5. (12 points) Let $\mathcal{W} = \text{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$ and $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Use the standard dot product to answer these questions:

   (a) (6 points) Find the orthogonal complement, $\mathcal{W}^\perp$.

   (b) (6 points) Find $\mathbf{w} \in \mathcal{W}$ and $\mathbf{z} \in \mathcal{W}^\perp$ such that $\mathbf{x} = \mathbf{w} + \mathbf{z}$.

Solution: (a) Let the spanning vectors of $\mathcal{W}$ be $\mathbf{w}_1$ and $\mathbf{w}_2$. We need to find all vectors $\mathbf{z}$ such that

$$\mathbf{w}_1^T \mathbf{z} = 0$$
$$\mathbf{w}_2^T \mathbf{z} = 0$$

This produces a homogeneous linear system of equations with a coefficient matrix

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix}$$

We need to find the kernel of A, so we row reduce:

$$A \to \begin{pmatrix} -1 & 1 & 1 \\ 0 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Setting the free variable to 1 and solving for the other two gives the vector $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Every multiple of this vector is orthogonal to the two basis vectors of $\mathcal{W}$, so we have

$$\mathcal{W}^\perp = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(b) Since we have basis vectors for both $\mathcal{W}$ and $\mathcal{W}^\perp$ we can calculate either $\mathbf{w}$ or $\mathbf{z}$ directly and then find the other via subtraction. Since $\mathcal{W}^\perp$ only has one basis vector, its simpler to compute $\mathbf{z}$ directly:

$$\mathbf{x}^T \mathbf{a} = 1 + 0 + 1 = 2$$
$$\mathbf{a}^T \mathbf{a} = 1 + 0 + 1 = 2$$
$$\mathbf{z} = \frac{\mathbf{x}^T \mathbf{a}}{\mathbf{a}^T \mathbf{a}} \mathbf{a} = \frac{2}{2} \mathbf{a}$$
$$\mathbf{z} = \mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Now we use $\mathbf{x} = \mathbf{w} + \mathbf{z}$ to find $\mathbf{w}$:

$$\mathbf{w} = \mathbf{x} - \mathbf{z} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$
$$\mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

6. (18 points) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 1 & 3 \end{pmatrix}$.

   (a) (2 points) Are the columns of A an orthogonal basis for img A using the dot product?

   (b) (16 points) Find a QR factorization of A.

Solution: (a) No. Writing the columns of A as $\mathbf{a}_i$ we have

$$\begin{aligned} \mathbf{a}_1^T\mathbf{a}_2 &= 1 - 2 + 1 = 0 \\ \mathbf{a}_1^T\mathbf{a}_3 &= 1 + 2 + 3 = 6 \neq 0 \\ \mathbf{a}_2^T\mathbf{a}_3 &= 1 - 4 + 3 = 0 \end{aligned}$$

Since the inner products are not **all** 0, they are not an orthogonal set, so aren't an orthogonal basis.

(b) We begin by finding r_{11} and normalizing $\mathbf{a}_1$ to get $\mathbf{u}_1$:

$$\begin{aligned} r_{11} &= ||\mathbf{a}_1|| = \sqrt{3} \\ \mathbf{u}_1 &= \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

Next we calculate r_{12} and r_{22} to get $\mathbf{u}_2$:

$$\begin{aligned} r_{12} &= \mathbf{u}_1^T\mathbf{a}_2 = \frac{1}{\sqrt{3}}\mathbf{a}_1^T\mathbf{a}_2 = 0 \\ r_{22} &= \sqrt{\mathbf{a}_2^T\mathbf{a}_2 - r_{12}^2} = \sqrt{1^2 + (-2)^2 + 1^2 - 0^2} = \sqrt{6} \\ \mathbf{u}_2 &= \frac{1}{\sqrt{6}}(\mathbf{a}_2 - r_{12}\mathbf{u}_1) = \frac{1}{\sqrt{6}}\mathbf{a}_2 \\ &= \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \end{aligned}$$

Finally, we calculate r_{13}, r_{23}, and r_{33} to get $\mathbf{u}_3$:

$$\begin{aligned} r_{13} &= \mathbf{u}_1^T\mathbf{a}_3 = \frac{1}{\sqrt{3}}\mathbf{a}_1^T\mathbf{a}_3 = \frac{1}{\sqrt{3}}(1 + 2 + 3) = \frac{6}{\sqrt{3}} = 2\sqrt{3} \\ r_{23} &= \mathbf{u}_2^T\mathbf{a}_3 = \frac{1}{\sqrt{6}}\mathbf{a}_2^T\mathbf{a}_3 = 0 \\ r_{33} &= \sqrt{\mathbf{a}_3^T\mathbf{a}_3 - r_{13}^2 - r_{23}^2} = \sqrt{1^2 + 2^2 + 3^2 - (2\sqrt{3})^2 - 0^2} \\ &= \sqrt{14 - 12} = \sqrt{2} \\ \mathbf{u}_3 &= \frac{1}{\sqrt{2}}(\mathbf{a}_3 - r_{13}\mathbf{u}_1 - r_{23}\mathbf{u}_2) = \frac{1}{\sqrt{2}}(\mathbf{a}_3 - 2\sqrt{3}\mathbf{u}_1) \\ &= \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 2\sqrt{3}\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}\right) \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

Creating Q from the unit vectors and the R matrix from the calculated coefficients gives

$$A = QR = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 2\sqrt{3} \\ 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$