Write your name below. This exam is worth 100 points and has 6 problems. You must show your work to receive full credit on the problems. You are allowed to use one side of an 8.5x11 page of notes. You cannot collaborate on the exam or seek outside help, nor can you use the recorded lectures, a calculator, any computational software, or material you find online.

Name:

1. (24 points, 8 each) In each of the following problems, either show that the statement is always true or provide a counterexample that shows it is false.

   (a) The $p = 1$ norm on $\mathbb{R}^2$, $||\cdot||_1$, does not have an associated inner product.

   (b) Let $K = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$, which factors into $\begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$. The vector product defined as $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T K \overline{\mathbf{y}}$ is an inner product on $\mathbb{C}^2$.

   (c) For every matrix A, $||A||_\infty \leq ||A||_1$.

2. (14 points) You have calculated the REF of matrix A and found that

$$A = LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 \\ 3 & -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 7 & 0 & 1 & 1 \\ 0 & 0 & 1 & 8 & 6 & 4 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

(a) (5 points) Find a basis for $\ker A$.

(b) (5 points) Is the vector $\begin{pmatrix} 2 & 1 & 8 & 8 & 1 & -2 \end{pmatrix}^T$ in the coimage of A?

(c) (4 points) What are the dimensions of all 4 fundamental subspaces of A?

3. (18 points) Let $K = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{pmatrix}$.

   (a) (6 points) What is the quadratic form $q(\mathbf{x}) = \mathbf{x}^T K \mathbf{x}$?

   (b) (10 points) Find the LDL^T factorization of K.

   (c) (2 points) Is the vector product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T K \mathbf{y}$ an inner product on $\mathbb{R}^3$?

4. (14 points) The following questions refer to the set of polynomials

$$\left\{x^2+1,\ x,\ x^2-1\right\}$$

(a) (4 points) Show that these polynomials are a basis for $\mathcal{P}^{(2)}$, the vector space of polynomials with degree ≤ 2.

(b) (5 points) Using the inner product $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)dx$, are these an orthogonal basis? (Hint: You may use symmetry to simplify your calculations.)

(c) (5 points) If the basis is not orthogonal, use the Gram-Schmidt procedure to find an orthogonal basis using the inner product from part (b).

5. (12 points) Let $\mathcal{W} = \text{Span}\left\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\right\}$ and $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Use the standard dot product to answer these questions:

   (a) (6 points) Find the orthogonal complement, $\mathcal{W}^{\perp}$.

   (b) (6 points) Find $\mathbf{w} \in \mathcal{W}$ and $\mathbf{z} \in \mathcal{W}^{\perp}$ such that $\mathbf{x} = \mathbf{w} + \mathbf{z}$.

6. (18 points) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 1 & 3 \end{pmatrix}$.

   (a) (2 points) Are the columns of A an orthogonal basis for img A using the dot product?

   (b) (16 points) Find a QR factorization of A.