1. The following are unrelated

 (a) (4 pts) Write the following sum in sigma notation

$$2 - 4 + 6 - 8 + ... - 20$$

Solution:
We notice that the these are all even numbers with alternating sign. Hence,

$$2 - 4 + 6 - 8 + ... - 20 = \boxed{\sum_{i=1}^{10} (-1)^{i+1}(2i)}$$

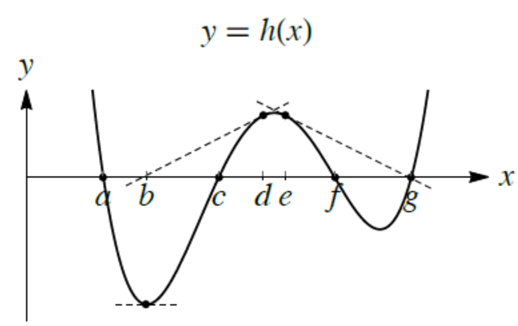
 (b) (10 pts) Express the following definite integral as a limit of Riemann Sums using the right endpoints. $\boxed{\textbf{DO NOT}}$ evaluate the limit.

$$\int_0^3 (x^2 + 3)\, dx$$

Solution:
Here $f(x) = (x^2 + 3), a = 0, b = 3$. Also, $\Delta x = \dfrac{3-0}{n} = \dfrac{3}{n}$. Hence, using the formula for the Riemann sum with right endpoints,

$$\begin{aligned}\int_0^3 (x^2+3)\, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(a + i\Delta x)\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{9i^2}{n^2} + 3\right)\left(\frac{3}{n}\right) \\ &= \boxed{\left(\frac{3}{n}\right) \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{9i^2}{n^2} + 3\right)}\end{aligned}$$

 (c) (6 pts) Suppose Newton's Method is used to find a root of the function $h(x)$. The tangent lines at $x = b, d$ and e are shown. (No explanation is necessary for the following questions.)



  i. For which of the initial approximations $x_1 = a, b, c, d, e, f, g$ will Newton's Method fail to find a root? List or circle all that apply.
  Solution:
  $\boxed{b, d}$

  ii. Which of the initial approximations $x_1 = b, d, e, g$ will lead to a second approximation x_2 that equals one of the roots of h? List or circle all that apply.
  Solution:
  $\boxed{e, g}$

2. (20 pts) The acceleration of a particle as a function of time t is given by $a(t) = \cos(\pi t)\ m/s$. Find the position of the particle as a function of time t if the velocity and position at $t = 1\ s$ are $1\ m/s$ and $\frac{1}{\pi^2}\ m$, respectively.
Solution:

$$\begin{aligned} v(t) &= \int a(t)\, dt \\ &= \int \cos(\pi t)\, dt \\ &= -\frac{\sin(\pi t)}{\pi} + C \end{aligned}$$

Plugging in the initial conditions

$$\begin{aligned} v(1) &= 1 \\ -\frac{\sin(\pi)}{\pi} + C &= 1 \\ C &= 1 \end{aligned}$$

Which means $v(t) = -\dfrac{\sin(\pi t)}{\pi} + 1$. Now,

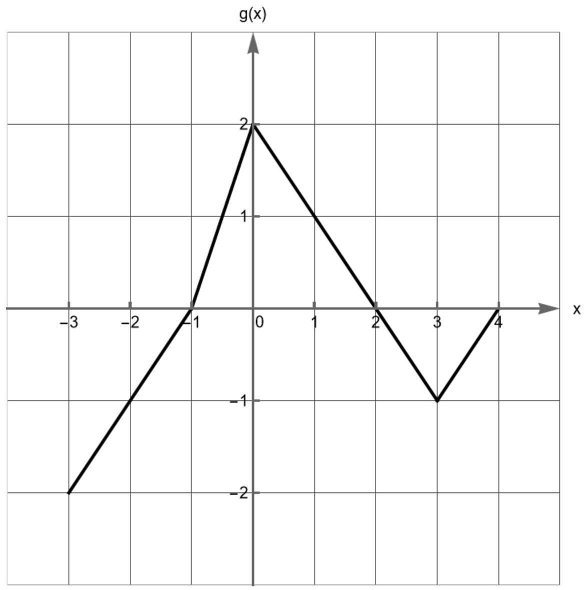
$$\begin{aligned} x(t) &= \int v(t)\, dt \\ &= \int \left(-\frac{\sin(\pi t)}{\pi} + 1\right) dt \\ &= -\frac{\cos(\pi t)}{\pi^2} + t + C_1 \end{aligned}$$

Plugging in the initial conditions

$$\begin{aligned} x(1) &= \frac{1}{\pi^2} \\ -\frac{\cos(\pi)}{\pi^2} + 1 + C_1 &= \frac{1}{\pi^2} \\ \frac{1}{\pi^2} + 1 + C_1 &= \frac{1}{\pi^2} \\ C_1 &= -1 \end{aligned}$$

Hence, $\boxed{x(t) = \left(-\dfrac{\cos(\pi t)}{\pi^2} + t - 1\right)\ m}$

3. The function f is defined as $f(x) = \int_{-3}^{x} g(t)\, dt$ where the graph of g is shown below



(a) (10 pts) Find all the values of $x \in (-3, 4)$ where f has a point of inflection. Show your work.
Solution:
Using the Fundamental Theorem of Calculus, we have

$$f(x) = \int_{-3}^{x} g(t)\, dt$$
$$f'(x) = g(x)$$
$$f''(x) = g'(x)$$

Looking at the graph of g, we see that $g'(x)$ is undefined as $x = -1$, $x = 1$ and $x = 3$.
At $x = -1$, $g'(x)$ doesn't change sign. However, at $x = 0$ and $x = 3$, $g'(x)$ changes sign. This means $f''(x)$ is undefined at those points, and $f''(x)$ is changing sign at $x = 0$ and $x = 3$. Hence f has a point of inflection at $\boxed{x = 0 \text{ and } x = 3}$

(b) (10 pts) Find the equation of the tangent line to the graph of f at $x = 1$.

Solution:
First, we calculate $f(1) = \int_{-3}^{1} g(t)\, dt$ as the net area under the graph, by splitting it into 2 triangles and a trapezoid.

$$\begin{aligned} f(2) &= -\frac{1}{2}(2)(2) + \frac{1}{2}(1)(2) + \frac{1}{2}(2+1)(1) \\ &= -2 + 1 + \frac{3}{2} \\ &= \frac{1}{2} \end{aligned}$$

Also, $f'(1) = g(1) = 1$. Hence the equation of tangent line at $x = 1$ is given by
$\left(y - \frac{1}{2}\right) = 1(x - 1)$, or, $\boxed{y = x - \frac{1}{2}}$

4. (10 pts) Let f be an odd function and g be an even function with the following properties:

   (i) $\int_{-1}^{0} f(x)\,dx = 2$

   (ii) $\int_{0}^{3} f(x)\,dx = 7$

   (iii) $\int_{1}^{3} g(x)\,dx = -3$

   Evaluate

$$\int_{-3}^{-1} (2f(x) - 4g(x))\ dx$$

Solution:

f is an odd function. Hence, $\int_{-3}^{0} f(x)\,dx = -\int_{0}^{3} f(x)\,dx = -7$. Using this, we can write

$$\begin{aligned}\int_{-3}^{-1} f(x)\,dx &= \int_{-3}^{0} f(x)\,dx + \int_{0}^{-1} f(x)\,dx \\ &= \int_{-3}^{0} f(x)\,dx - \int_{-1}^{0} f(x)\,dx \\ &= (-7) - (2) \\ &= -9\end{aligned}$$

Also, g is an even function. Hence, $\int_{-3}^{-1} g(x)\,dx = \int_{1}^{3} g(x)\,dx = -3$

Thus, we have

$$\begin{aligned}\int_{-3}^{-1} (2f(x) - 4g(x))\ dx &= 2\int_{-3}^{-1} f(x)\,dx - 4\int_{-3}^{-1} g(x)\,dx \\ &= 2(-9) - 4(-3) \\ &= \boxed{-6}\end{aligned}$$

5. Evaluate the following integrals by any method you like

   (a) (10 pts) $\int t^2\sqrt{t-4}\,dt$

   Solution:

   We employ the substitution $u = t - 4, du = dt$. Hence the given integral reduces to

$$\begin{aligned}\int t^2\sqrt{t-4}\,dt &= \int (u+4)^2\sqrt{u}\,du \\ &= \int (u^2 + 8u + 16)u^{\frac{1}{2}}\ du \\ &= \int u^{\frac{5}{2}}\ du + 8\int u^{\frac{3}{2}}\ du + 16\int u^{\frac{1}{2}}\ du \\ &= \frac{2}{7}u^{\frac{7}{2}} + \frac{16}{5}u^{\frac{5}{2}} + \frac{32}{3}u^{\frac{2}{3}} + C \\ &= \boxed{\frac{2}{7}(t-4)^{\frac{7}{2}} + \frac{16}{5}(t-4)^{\frac{5}{2}} + \frac{32}{3}(t-4)^{\frac{3}{2}} + C}\end{aligned}$$

(b) (10 pts) $\int_{-1}^{3} |x-1|\,dx$

Solution:

We know that

$$|x-1| = \begin{cases} (x-1) & \text{if } x \geq 1, \\ -(x-1) & \text{if } x < 1 \end{cases}$$

Hence, we can rewrite the integral as

$$\begin{aligned} \int_{-1}^{3} |x-1|\,dx &= \int_{-1}^{1} |x-1|\,dx + \int_{1}^{3} |x-1|\,dx \\ &= \int_{-1}^{1} -(x-1)\,dx + \int_{1}^{3} (x-1)\,dx \\ &= \int_{-1}^{1} (1-x)\,dx + \int_{1}^{3} (x-1)\,dx \\ &= \left[x - \frac{x^2}{2}\right]_{-1}^{1} + \left[\frac{x^2}{2} - x\right]_{1}^{3} \\ &= \left(1 - \frac{1}{2}\right) - \left(-1 - \frac{1}{2}\right) + \left(\frac{9}{2} - 3\right) - \left(\frac{1}{2} - 1\right) \\ &= \boxed{4} \end{aligned}$$

(c) (10 pts) $\int_{-362}^{362} \sin\left(x^{99}\right)\,dx + \int_{0}^{\frac{\pi}{2}} \sin(\sin x)\cos x\,dx$

Solution:

The 1st integral evaluates to 0, because it is an integral (net area under the graph) of an odd function with symmetric bounds.

We evaluate the 2nd integral using the substitution $u = \sin x \Rightarrow du = \cos x dx$. Also, $x = 0 \Rightarrow u = 0$ and $x = \frac{\pi}{2} \Rightarrow u = 1$. Hence the integral reduces to

$$\begin{aligned} \int_{0}^{\frac{\pi}{2}} \sin(\sin x)\cos x\,dx &= \int_{0}^{1} \sin(u)\,du \\ &= \left[-\cos u\right]_{0}^{1} \\ &= -\left(\cos(1) - \cos(0)\right) \\ &= \boxed{1 - \cos(1)} \end{aligned}$$