1. [2360/070626 (24 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a two-column table (letter - answer) completely separate from any work you do to arrive at the answer.

(a) The general solution of the constant coefficient linear homogeneous differential equation whose characteristic equation is $\left(r^2 - 2r + 4\right)^3 = 0$ is $y(t) = e^t\left[\left(c_1 + c_2 t + c_3 t^2\right)\cos\sqrt{3}t + \left(c_4 + c_5 t + c_6 t^2\right)\sin\sqrt{3}t\right]$

(b) Let $x(t) = 3e^{-2t}\cos t - 2e^{-2t}\sin t + \sqrt{2}\cos(5t - \delta)$ be the equation of motion of an harmonic oscillator. The time-varying amplitude of the transient solution is $\sqrt{2}e^{-2t}$.

(c) Let $x(t)$ be the displacement of an undamped harmonic oscillator subjected to a nonzero driving force, $f(t)$, that approaches zero as $t \to \infty$. Then $\lim\limits_{t\to\infty} x(t) = 0$.

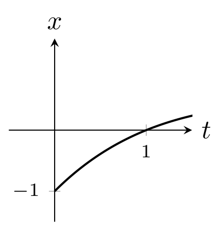
(d) The differential equation $3\ddot{u} + 2\cos u = 5$ describes a conservative system.

(e) The differential equation $y''' - y = 1$ can be written as a system of differential equations in the form $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

(f) $\displaystyle\int_0^\infty e^{-st}\cos\left(\pi^2 t\right)\,\mathrm{d}t = \frac{\pi^2}{s^2 + \pi^2}$

(g) Every nontrivial solution to a constant coefficient linear homogeneous differential equation whose characteristic roots are $1 \pm 3i$ exhibits oscillations with growing amplitude.

(h) A portion of the equation of motion for an unforced mass/spring system is shown in the following figure. From the graph, the mass was released from rest 1 unit to the left of its equilibrium position.



SOLUTION:

(a) **TRUE** The roots of the characteristic equation are $1 \pm i\sqrt{3}$, each with multiplicity 3. A basis for the solution space is

$$\left\{e^t\cos\sqrt{3}t, te^t\cos\sqrt{3}t, t^2e^t\cos\sqrt{3}t, e^t\sin\sqrt{3}t, te^t\sin\sqrt{3}t, t^2e^t\sin\sqrt{3}t\right\}$$

so the general solution is an element in the span of the basis vectors.

(b) **FALSE** The transient solution is $3e^{-2t}\cos t - 2e^{-2t}\sin t$, the amplitude of which is $\sqrt{\left(3e^{-2t}\right)^2 + \left(-2e^{-2t}\right)^2} = \sqrt{13}e^{-2t}$

(c) **FALSE** Even though the forcing vanishes as t goes to infinity, the oscillations do not since there is no damping (the homogeneous solution consisting of sines and/or cosines remains)

(d) **TRUE** It can be written in the form $m\ddot{u} + V'(u) = 0$ where $m = 3$ and $V'(u) = 2\cos u - 5$

(e) **FALSE** Let $x_1 = y, x_2 = y', x_3 = y''$. Then

$$x_1' = y' = x_2$$
$$x_2' = y'' = x_3$$
$$x_3' = y''' = 1 + y = 1 + x_1$$

With $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ we have

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{f}}$$

(f) **FALSE** $\int_0^\infty e^{-st}\cos(\pi^2 t)\,\mathrm{d}t = \mathscr{L}\left\{\cos\left(\pi^2 t\right)\right\} = \dfrac{s}{s^2+\pi^4}$

(g) **TRUE** Solutions are of the form $e^t\left(c_1\cos 3t + c_2\sin 3t\right)$ which oscillate with increasing amplitude

(h) **FALSE** The slope of the graph at $t=0$ is positive meaning the initial velocity was positive, not zero

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2. [2360/070626 (23 pts)] Consider the linear operator $L(\vec{\mathbf{y}}) = (1-\ln t)y'' + \dfrac{y'}{t} - \dfrac{y}{t^2}$. Assume $t > e$.

   (a) (8 pts) Show that the solution space of $L(\vec{\mathbf{y}}) = 0$ equals span $\{t, \ln t\}$.

   (b) (15 pts) Find the general solution of $L(\vec{\mathbf{y}}) = \dfrac{(1-\ln t)^2}{t}$.

SOLUTION:

(a) Begin by checking that t and $\ln t$ are solutions.

$$(1-\ln t)t'' + \frac{t'}{t} - \frac{t}{t^2} = (1-\ln t)(0) + \frac{1}{t} - \frac{1}{t} = 0 \quad \checkmark$$

$$(1-\ln t)(\ln t)'' + \frac{(\ln t)'}{t} - \frac{\ln t}{t^2} = -\frac{1}{t^2} + \frac{\ln t}{t^2} + \frac{1}{t^2} - \frac{\ln t}{t^2} = 0 \quad \checkmark$$

Now check for linear independence.

$$W[t, \ln t](t) = \begin{vmatrix} t & \ln t \\ 1 & 1/t \end{vmatrix} = 1 - \ln t$$

This is never 0 on the interval $t > e$ implying that the solutions are linearly independent. Since this is a second order linear homogeneous equation, $\{t, \ln t\}$ forms a basis for the two-dimensional solution space. The solution space is therefore given by span $\{t, \ln t\}$.

(b) We must use variation of parameters to solve $(1-\ln t)y'' + \dfrac{y'}{t} - \dfrac{y}{t^2} = \dfrac{(1-\ln t)^2}{t}$. Rewriting this in standard form gives

$$y'' + \frac{y'}{t(1-\ln t)} - \frac{y}{t^2(1-\ln t)} = \frac{1-\ln t}{t}$$

With $y_1 = t$ and $y_2 = \ln t$ the particular solution will be given by $y_p(t) = v_1(t)t + v_2(t)\ln t$ where

$$v_1 = \int \frac{-\ln t\left(\frac{1-\ln t}{t}\right)}{1-\ln t}\,\mathrm{d}t = -\int \frac{\ln t}{t}\,\mathrm{d}t \overset{u=\ln t}{=} -\frac{1}{2}(\ln t)^2$$

$$v_2 = \int \frac{t\left(\frac{1-\ln t}{t}\right)}{1-\ln t}\,\mathrm{d}t = \int \mathrm{d}t = t$$

$$y_p = -\frac{1}{2}t(\ln t)^2 + t\ln t = t\ln t\left(1 - \frac{1}{2}\ln t\right)$$

Applying the Nonhomogeneous Principle, the general solution is $y(t) = c_1 t + c_2\ln t + t\ln t\left(1 - \frac{1}{2}\ln t\right)$.

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3. [2360/070626 (22 pts)] You have decided to give your twin niece and nephew a sweet new toy for their upcoming birthday. It consists of a 1-kg bucket hooked to a spring which is then attached horizontally to a wall. The restoring/spring constant is $\frac{1}{4}$ N/m and the device is hooked up so that the damping force is equal to 4 times the instantaneous velocity of the bucket. A bag of 1-kg stones is included with the toy. Hint: There are no irrational numbers involved in this problem.

   (a) (15 pts) Upon seeing the toy, the kids put 6 stones into the 1-kg bucket, pull the bucket 1 meter to the right of the equilibrium position and impart a leftward velocity to it of one-half meter per second. How many times will the bucket pass through the equilibrium position? Justify your answer.

   (b) (4 pts) What is the smallest number of stones that need to be in the bucket so that the kids can have the excitement of watching the bucket oscillate? Justify your answer.

   (c) (3 pts) The kids really want a thrill and ask you to try to make the toy exhibit resonance. So you have them put 15 stones into the bucket and place the toy onto a table which provides a driving force to the toy given by $f(t) = \cos \frac{1}{8}t$. Will the kids get a thrill? Justify your answer.

SOLUTION:

(a) The initial value problem governing the motion is $7\ddot{x} + 4\dot{x} + \dfrac{1}{4}x = 0,\ x(0) = 1,\ \dot{x}(0) = -\dfrac{1}{2}$. The characteristic equation is

$$7r^2 + 4r + \frac{1}{4} = 0$$

$$r = \frac{-4 \pm \sqrt{4^2 - 4(7)\left(\frac{1}{4}\right)}}{2(7)} = \frac{-4 \pm 3}{14} = -\frac{1}{2}, -\frac{1}{14}$$

so the general solution is $x(t) = c_1 e^{-t/2} + c_2 e^{-t/14}$. Applying the initial conditions yields the linear system

$$c_1 + c_2 = 1$$

$$-\frac{1}{2}c_1 - \frac{1}{14}c_2 = -\frac{1}{2}$$

$$c_1 = \frac{\begin{vmatrix} 1 & 1 \\ -\frac{1}{2} & -\frac{1}{14} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -\frac{1}{2} & -\frac{1}{14} \end{vmatrix}} = 1 \qquad c_2 = \frac{\begin{vmatrix} 1 & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -\frac{1}{2} & -\frac{1}{14} \end{vmatrix}} = 0$$

So the solution to the initial value problem is $x(t) = e^{-t/2}$ which is never zero, implying that the bucket never passes through the equilibrium position.

(b) We need $b^2 - 4mk = 16 - 4(m)\left(\dfrac{1}{4}\right) = 16 - m < 0$ which will occur first here when $m = 17$. At least sixteen stones need to be in the bucket since the bucket itself has a mass of 1 kg.

(c) No. Even though $\omega_0 = \frac{1}{8}$ matches the forcing frequency, damped oscillators never experience resonance.

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4. [2360/070626 (15 pts)] Use the Method of Undetermined Coefficients to find the general solution of $\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} = -40\cos^2 t$. Hint: $2\cos^2 t = 1 + \cos 2t$.

SOLUTION:

Using the hint, rewrite the equation as $y'' - y' = -20(2\cos^2 t) = -20(1 + \cos 2t) = -20 - 20\cos 2t$. Find the solution to the associated homogeneous equation.

$$y_h'' - y_h' = 0$$

$$r^2 - r = r(r-1) = 0$$

$$r = 0, 1$$

$$y_h = c_1 + c_2 e^t$$

Find the particular solution. Note that a constant is a solution to the associated homogeneous equation.

$$\begin{aligned} y_p &= At + B\cos 2t + C\sin 2t \\ y_p' &= A - 2B\sin 2t + 2C\cos 2t \\ y_p'' &= -4B\cos 2t - 4C\sin 2t \end{aligned}$$

$$\begin{aligned} y_p'' - y_p' &= -4B\cos 2t - 4C\sin 2t - (A - 2B\sin 2t + 2C\cos 2t) \\ &= -A + (-4B - 2C)\cos 2t + (2B - 4C)\sin 2t = -20 - 20\cos 2t \end{aligned}$$

$$-A = -20 \implies A = 20$$

$$-4B - 2C = -20 \implies 2B + C = 10$$

$$2B - 4C = 0 \implies B - 2C = 0$$

$$\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} B \\ C \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$B = \frac{\begin{vmatrix} 10 & 1 \\ 0 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{-20}{-5} = 4 \qquad C = \frac{\begin{vmatrix} 2 & 10 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{-10}{-5} = 2$$

$$y_p = 20t + 4\cos 2t + 2\sin 2t$$

Application of the Nonhomogeneous Principle yields the general solution of $y = y_h + y_p = c_1 + c_2e^t + 20t + 4\cos 2t + 2\sin 2t$. ■

5. [2360/070626 (16 pts)] Evaluate the following.

   (a) (6 pts) $\mathscr{L}\left\{\left(t^2+3\right)^2 e^{-7t}\right\}$

   (b) (10 pts) $\mathscr{L}^{-1}\left\{\dfrac{5s-12}{s^2-6s+13}\right\}$

SOLUTION:

(a)

$$\begin{aligned} \mathscr{L}\left\{\left(t^2+3\right)^2 e^{-7t}\right\} &= \mathscr{L}\left\{\left(t^4+6t^2+9\right) e^{-7t}\right\} \\ &= \mathscr{L}\left\{t^4e^{-7t}\right\} + 6\mathscr{L}\left\{t^2e^{-7t}\right\} + 9\mathscr{L}\left\{e^{-7t}\right\} \\ &= \mathscr{L}\left\{t^4\right\}\big|_{s\to s+7} + 6\mathscr{L}\left\{t^2\right\}\big|_{s\to s+7} + \frac{9}{s+7} \\ &= \left.\frac{4!}{s^5}\right|_{s\to s+7} + 6\left.\frac{2!}{s^3}\right|_{s\to s+7} + \frac{9}{s+7} \\ &= \frac{24}{(s+7)^5} + \frac{12}{(s+7)^3} + \frac{9}{s+7} \end{aligned}$$

(b)

$$\frac{5s-12}{s^2-6s+13} = \frac{5s-12}{s^2-6s+13+9-9} = \frac{5s-12}{(s-3)^2+4} = \frac{5s-12}{(s-3)^2+2^2}$$

We need the numerator to have the form $5s - 12 = A(s-3) + B(2)$. Thus $A = 5$ and $-15 + 2B = -12 \implies B = 3/2$. Then

$$\frac{5s-12}{s^2-6s+13} = \frac{5(s-3)}{(s-3)^2+2^2} + \frac{\left(\frac{3}{2}\right)(2)}{(s-3)^2+2^2}$$

and

$$
\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{5s-12}{s^2-6s+13}\right\} &= \mathscr{L}^{-1}\left\{\frac{5(s-3)}{(s-3)^2+2^2} + \frac{\left(\frac{3}{2}\right)(3)}{(s-3)^2+2^2}\right\} \\
&= 5\mathscr{L}^{-1}\left\{\frac{(s-3)}{(s-3)^2+2^2}\right\} + \frac{3}{2}\mathscr{L}^{-1}\left\{\frac{2}{(s-3)^2+2^2}\right\} \\
&= 5e^{3t}\cos 2t + \frac{3}{2}e^{3t}\sin 2t
\end{aligned}
$$

$\blacksquare$