1. [2350/070626 (15 pts)] Evaluate $\displaystyle\int_0^1 \int_{\sqrt[3]{z}}^1 \int_0^{\ln 3} \frac{\pi e^{2x}\sin(\pi y^2)}{y^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

SOLUTION:

$$\int_0^1 \int_{\sqrt[3]{z}}^1 \int_0^{\ln 3} \frac{\pi e^{2x}\sin\left(\pi y^2\right)}{y^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^1 \int_{\sqrt[3]{z}}^1 \frac{\pi\sin(\pi y^2)}{y^2}\frac{1}{2}e^{2x}\Big|_0^{\ln 3}\,\mathrm{d}y\,\mathrm{d}z = 4\pi\int_0^1 \int_{\sqrt[3]{z}}^1 \frac{\sin\left(\pi y^2\right)}{y^2}\mathrm{d}y\,\mathrm{d}z \qquad \text{(switch order)}$$

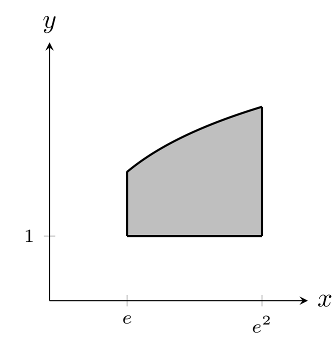
$$= 4\pi\int_0^1 \int_0^{y^3} \frac{\sin\left(\pi y^2\right)}{y^2}\mathrm{d}z\,\mathrm{d}y = 4\pi\int_0^1 \frac{\sin\left(\pi y^2\right)}{y^2}z\Big|_0^{y^3}\,\mathrm{d}y = 4\pi\int_0^1 y\sin\left(\pi y^2\right)\,\mathrm{d}y \overset{u=\pi y^2}{=} 4\pi\int_0^{\pi}\frac{1}{2\pi}\sin u\,\mathrm{d}u = 4$$

■

2. [2350/070626 (15 pts)] A thin metal plate (lamina) is placed on the xy-plane. It's shape is shown in the figure below with the curved upper portion described by $y = 1 + \ln x$. The plate's density is $\rho(x, y) = \sqrt{xy^3}$. Carefully follow the directions below and simplify your answers.

   (a) Set up, but **do not evaluate**, an appropriate double integral using the order $\mathrm{d}x\,\mathrm{d}y$ that will give the lamina's moment of inertia about the x-axis.

   (b) Set up, but **do not evaluate**, an appropriate double integral using the order $\mathrm{d}y\,\mathrm{d}x$ that will give the lamina's moment of inertia about the y-axis.



SOLUTION:

(a)

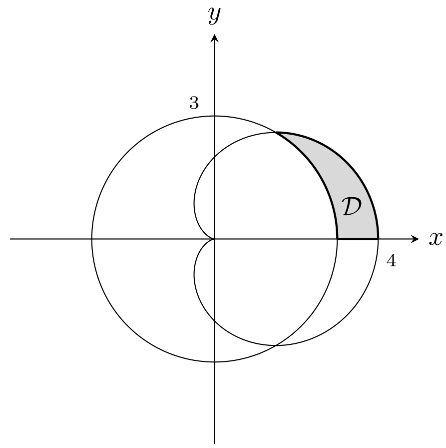
$$I_x = \iint_{\mathcal{D}} y^2\rho(x,y)\,\mathrm{d}A = \iint_{\mathcal{D}} y^2\sqrt{xy^3}\,\mathrm{d}A = \int_1^2\int_e^{e^2} x^{1/2}y^{7/2}\,\mathrm{d}x\,\mathrm{d}y + \int_2^3\int_{e^{y-1}}^{e^2} x^{1/2}y^{7/2}\,\mathrm{d}x\,\mathrm{d}y$$

(b)

$$I_y = \iint_{\mathcal{D}} x^2\rho(x,y)\,\mathrm{d}A = \iint_{\mathcal{D}} x^2\sqrt{xy^3}\,\mathrm{d}A = \int_e^{e^2}\int_1^{1+\ln x} x^{5/2}y^{3/2}\,\mathrm{d}y\,\mathrm{d}x$$

■

3. [2350/070626 (23 pts)] You have built a new clubhouse whose floor (lying in the xy-plane) is in the shape of the shaded region $\mathcal{D}$ in the accompanying figure. The curves shown in the figure are $x^2 + y^2 = 9$ and $r = 2(1 + \cos\theta)$. The roof of the clubhouse is given by $f(x, y) = \dfrac{8y}{x^2 + y^2}$. Determine the volume of the clubhouse by evaluating an appropriate double integral.



SOLUTION:

We need to use polar coordinates. The circle's polar coordinates equation is $r = 3$. The height of the roof in terms of r and θ is

$$f(r,\theta) = \frac{8r\sin\theta}{r^2} = \frac{8\sin\theta}{r}$$

We also need to find the intersection point of the two curves:

$$3 = 2(1+\cos\theta)$$

$$\frac{1}{2} = \cos\theta \implies \theta = \frac{\pi}{3}$$

$$\text{Volume} = \iint_{\mathcal{D}} f(x,y)\,\mathrm{d}A = \int_0^{\pi/3}\int_3^{2(1+\cos\theta)} \frac{8\sin\theta}{r} r\,\mathrm{d}r\,\mathrm{d}\theta = 8\int_0^{\pi/3}\sin\theta\int_3^{2(1+\cos\theta)}\mathrm{d}r\,\mathrm{d}\theta$$

$$= 8\int_0^{\pi/3}\sin\theta\, r\Big|_3^{2(1+\cos\theta)}\mathrm{d}\theta = 8\int_0^{\pi/3}\sin\theta\,(2\cos\theta - 1)\,\mathrm{d}\theta$$

$$= 8\int_0^{\pi/3}(\sin 2\theta - \sin\theta)\,\mathrm{d}\theta = 8\left(-\frac{1}{2}\cos 2\theta + \cos\theta\right)\Bigg|_0^{\pi/3}$$

$$= 8\left[\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) + \frac{1}{2} - \left(-\frac{1}{2}+1\right)\right] = \frac{8}{4} = 2$$

■

4. [2350/070626 (24 pts)] The charge density on a metal plate is given by $q(x,y) = 6x^2y^2$ coulombs per square meter. The plate occupies the first quadrant region bounded by $xy = 1$, $xy = 2$, $y/x = 1$, and $y/x = 4$. Find the total charge on the plate, including appropriate units in your answer.

SOLUTION:

Use the change of variables $u = xy$ and $v = y/x$. Then

$$u = xy, v = y/x \implies uv = y^2 \implies y = \sqrt{uv} = u^{1/2}v^{1/2} \quad \text{and} \quad u/v = x^2 \implies x = \sqrt{u/v} = u^{1/2}v^{-1/2}$$

(positive square roots since the plate is in the first quadrant)

$$J(u,v) = \begin{vmatrix} \frac{1}{2}u^{-1/2}v^{-1/2} & -\frac{1}{2}u^{1/2}v^{-3/2} \\ \frac{1}{2}u^{-1/2}v^{1/2} & \frac{1}{2}u^{1/2}v^{-1/2} \end{vmatrix} = \frac{1}{2v}$$

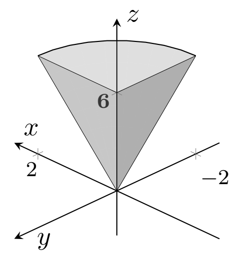
The region of integration is the rectangle $1 \le u \le 9$, $1 \le v \le 4$. Thus (we can eliminate the absolute value since $v > 0$)

$$\text{Charge} = \int_1^2\int_1^4 6u^2\left|\frac{1}{2v}\right|\mathrm{d}v\,\mathrm{d}u = \left(\int_1^2 3u^2\,\mathrm{d}u\right)\left(\int_1^4 \frac{1}{v}\,\mathrm{d}v\right) = \left(u^3\Big|_1^2\right)\left(\ln|v|\Big|_1^4\right) = 7\ln 4 \text{ coulombs}$$

■

5. [2350/070626 (23 pts)] Consider the solid region, $\mathcal{W}$, shown below, which is a portion of the cone $x^2 + y^2 - \left(\frac{z}{3}\right)^2 = 0$. Suppose that the density, $\delta(x, y, z)$, of the region's material at each point in the solid is equal to the distance between the point (x, y, z) and the origin. Set up, but **do not evaluate**, triple integral(s) using the order shown that will compute the mass of the solid. To receive full credit, you must use the correct bounds for the figure as shown (study it carefully), not bounds for an equivalent solid in a different octant.

(a) $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ (b) $\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$ (c) $\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$



SOLUTION:

$\mathcal{W}$ is the "inside" of the portion of the cone of radius 2 and height 6 with vertex at the origin that resides above Quadrant IV. The density can be written as $\delta(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

(a) Arbitrary y, z: x enters region at 0 and exits at $\sqrt{z^2/9 - y^2}$. Project onto yz-plane.
Arbitrary z: y enters projected region at $-z/3$ and exits at 0. Then sum up z from 0 to 6.

$$\text{Mass} = \iiint_{\mathcal{W}} \delta\,\mathrm{d}V = \int_0^6 \int_{-z/3}^0 \int_0^{\sqrt{z^2/9-y^2}} \sqrt{x^2 + y^2 + z^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

(b) Arbitrary r, θ: z enters region at $3r$ and exits at 6. Project onto the xy-plane.

Arbitrary θ: r enters projected region at 0 and exits at 2. Then sum up θ from $3\pi/2$ to 2π.

$$\text{Mass} = \iiint_{\mathcal{W}} \delta\,\mathrm{d}V = \int_{3\pi/2}^{2\pi} \int_0^2 \int_{3r}^6 r\sqrt{r^2 + z^2}\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

(c) Arbitrary θ, ϕ: ρ enters the region at 0 and exits at $6\sec\phi$. These rays are then summed from $\theta = 3\pi/2$ to $\theta = 2\pi$ and swept from $\phi = 0$ to $\phi = \tan^{-1}\frac{1}{3}$.

$$\text{Mass} = \iiint_{\mathcal{W}} \delta\,\mathrm{d}V = \int_0^{\tan^{-1}\frac{1}{3}} \int_{3\pi/2}^{2\pi} \int_0^{6\sec\phi} \rho^3 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$$

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