1. (30pts) Determine whether each of the following series converges or diverges. Justify your answers and name any test or theorem that you use.

   (a) (8pts) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2n^2+5}$

   Solution: Examine the limit of the terms:

$$\lim_{n\to\infty} \frac{n^2}{2n^2+5} = \lim_{n\to\infty} \frac{1}{2+5/n^2} = \frac{1}{2} \neq 0.$$

   Since the terms do not approach 0, the series **diverges** by the Test for Divergence.

   (b) (10pts) $\displaystyle\sum_{n=1}^{\infty} n\, e^{-n^2}$

   Solution: Let $f(x) = xe^{-x^2}$. This is positive and continuous for $x \geq 1$, and decreasing there since

$$f'(x) = e^{-x^2}\left(1-2x^2\right) < 0 \quad \text{for } x \geq 1.$$

   So the Integral Test applies. With the substitution $u = x^2$, $du = 2x\,dx$,

$$\int_1^{\infty} xe^{-x^2}\,dx = \lim_{t\to\infty} \frac{1}{2}\int_1^{t^2} e^{-u}\,du = \lim_{t\to\infty} \frac{1}{2}\Big[-e^{-u}\Big]_1^{t^2} = \frac{1}{2}\,e^{-1}.$$

   The integral converges to a finite value, so by the Integral Test the series **converges**.

   (c) (10pts) $\displaystyle\sum_{n=1}^{\infty} \frac{3n+1}{\sqrt{2n^5-n+4}}$

   Solution: For large n the terms behave like $\dfrac{3n}{\sqrt{2n^5}} = \dfrac{3}{\sqrt{2}} \cdot \dfrac{1}{n^{3/2}}$, so compare to the convergent p-series $\sum \dfrac{1}{n^{3/2}}$ ($p = \frac{3}{2} > 1$). Using the Limit Comparison Test with $b_n = \dfrac{1}{n^{3/2}}$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n+1}{\sqrt{2n^5-n+4}} \cdot n^{3/2} = \lim_{n\to\infty} \frac{3n^{5/2}+n^{3/2}}{\sqrt{2n^5-n+4}} = \frac{3}{\sqrt{2}} > 0.$$

   Since the limit is finite and positive and $\sum b_n$ converges, by the Limit Comparison Test the series **converges**.

2. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n-2}$.

   (a) (12pts) Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Fully justify your answer.

   Solution: Convergence (Alternating Series Test). The series is alternating with $b_n = \dfrac{1}{3n-2} > 0$. The two hypotheses of the Alternating Series Test hold:

   - $\{b_n\}$ is decreasing, since $3n-2$ increases with n, so $b_{n+1} = \dfrac{1}{3(n+1)-2} < \dfrac{1}{3n-2} = b_n$;
   - $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{3n-2} = 0.$

By the Alternating Series Test, the series converges.

Not absolute. Examine $\sum_{n=1}^{\infty} \frac{1}{3n-2}$. Since $3n-2 < 3n$ for all $n \geq 1$,

$$\frac{1}{3n-2} > \frac{1}{3n} > 0.$$

The series $\sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n}$ is a constant multiple of the divergent harmonic series, hence diverges.

By the Direct Comparison Test, $\sum \frac{1}{3n-2}$ diverges as well. Since the series converges but not absolutely, it is **conditionally convergent**.

(b) (8pts) How many terms of the series are needed to guarantee that the partial sum approximates the true sum with an error less than 0.01?

Solution: By the Alternating Series Estimation Theorem, using the partial sum s_n of the first n terms gives an error bounded by the next term:

$$|s - s_n| \leq b_{n+1} = \frac{1}{3(n+1)-2} = \frac{1}{3n+1}.$$

We guarantee an error less than 0.01 by requiring

$$\frac{1}{3n+1} < 0.01 \quad \Longrightarrow \quad 3n+1 > 100 \quad \Longrightarrow \quad n > 33.$$

Thus $n = 34$ terms are needed.

3. Consider the power series $f(x) = \sum_{n=1}^{\infty} \frac{n!}{3^n\,(n+1)!}\,(x-2)^n$.

(a) (10pts) Determine the radius of convergence for $f(x)$.

Solution: First simplify the coefficient: $\frac{n!}{(n+1)!} = \frac{1}{n+1}$, so

$$f(x) = \sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n(n+1)}.$$

Apply the Ratio Test with $a_n = \frac{(x-2)^n}{3^n(n+1)}$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(x-2)^{n+1}}{3^{n+1}(n+2)} \cdot \frac{3^n(n+1)}{(x-2)^n} = \frac{n+1}{3(n+2)}\,|x-2|.$$

Taking the limit,

$$L = \lim_{n\to\infty} \frac{n+1}{3(n+2)}\,|x-2| = \frac{1}{3}\,|x-2|.$$

The series converges absolutely when $L < 1$, i.e. $|x-2| < 3$, so the radius of convergence is $\boxed{R = 3}$.

(b) (10pts) Determine the interval of convergence for $f(x)$.

Solution: From part (a), the series converges for $|x-2| < 3$, i.e. $-1 < x < 5$. Check the endpoints.

At $x = 5$: $(x-2)^n = 3^n$, so the series becomes

$$\sum_{n=1}^{\infty} \frac{3^n}{3^n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n+1}.$$

Since $\frac{1}{n+1} > \frac{1}{2n} > 0$ and $\sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n}$ diverges, this series diverges by the Direct Comparison Test.

At $x = -1$**:** $(x-2)^n = (-3)^n$, so the series becomes

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{3^n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}.$$

This is alternating with $b_n = \frac{1}{n+1}$, which is positive, decreasing, and $\to 0$, so it converges by the Alternating Series Test.

Therefore the interval of convergence is $\boxed{[-1, 5)}$.

(c) (6pts) For the following questions, you may use inequalities, interval notation, or list all the values (or say "there are no such x-values").

i. For which x values is this series absolutely convergent?
ii. For which x-values is this series conditionally convergent?
iii. For which x-values is this series divergent?

Solution: At $x = -1$ the series converges (by the AST), but its absolute version is $\sum \frac{1}{n+1}$, which diverges—so $x = -1$ is conditionally convergent.

i. Absolutely convergent on $(-1, 5)$.
ii. Conditionally convergent at $x = -1$.
iii. Divergent on $(-\infty, -1) \cup [5, \infty)$.

4. Let $f(x) = \sin^{-1}(x)$. Recall that $f'(x) = \frac{1}{\sqrt{1-x^2}}$.

(a) (6pts) Find the Maclaurin series for $f'(x) = \frac{1}{\sqrt{1-x^2}}$.

Solution: Start from the binomial series $(1+u)^k = \sum_{n=0}^{\infty} \binom{k}{n} u^n$ with $k = -\frac{1}{2}$ and $u = -x^2$:

$$\left(1 - x^2\right)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n \binom{-1/2}{n} x^{2n}.$$

(b) (10pts) The binomial series you found in part (a) can be rewritten as

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} x^{2n}.$$

Verify that the series you found in part (a) and this series agree for $n = 0, 1,$ and 2.

Solution: Recall the generalized binomial coefficient

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!},$$

where the numerator has exactly n factors. Writing out the cases $n = 0, 1, 2, 3$:

$$\binom{k}{0} = \frac{1}{0!}, \qquad \binom{k}{1} = \frac{k}{1!}, \qquad \binom{k}{2} = \frac{k(k-1)}{2!}, \qquad \binom{k}{3} = \frac{k(k-1)(k-2)}{3!}.$$

(For $n = 0$ the numerator is an empty product, equal to 1.) Now substitute $k = -\frac{1}{2}$:

$$\binom{-1/2}{0} = \frac{1}{0!} = 1, \qquad \binom{-1/2}{1} = \frac{-\frac{1}{2}}{1!} = -\frac{1}{2},$$

$$\binom{-1/2}{2} = \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!} = \frac{3/4}{2} = \frac{3}{8}, \qquad \binom{-1/2}{3} = \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!} = \frac{-15/8}{6} = -\frac{5}{16}.$$

From part (a), the coefficient of x^{2n} is $(-1)^n \binom{-1/2}{n}$, so the first four terms are

$$\underbrace{(+1)(1)}_{n=0}\, x^0 + \underbrace{(-1)\left(-\tfrac{1}{2}\right)}_{n=1} x^2 + \underbrace{(+1)\left(\tfrac{3}{8}\right)}_{n=2} x^4 + \underbrace{(-1)\left(-\tfrac{5}{16}\right)}_{n=3} x^6 = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \cdots$$

Notice the $(-1)^n$ always cancels the sign, leaving positive coefficients $1,\ \frac{1}{2},\ \frac{3}{8},\ \frac{5}{16}, \ldots$

Now compare against the proposed closed form $\dfrac{(2n)!}{4^n (n!)^2}$ for the same values of n:

$$n = 0:\ \frac{0!}{4^0(0!)^2} = 1, \qquad n = 1:\ \frac{2!}{4^1(1!)^2} = \frac{2}{4} = \frac{1}{2},$$

$$n = 2:\ \frac{4!}{4^2(2!)^2} = \frac{24}{16 \cdot 4} = \frac{24}{64} = \frac{3}{8}, \qquad n = 3:\ \frac{6!}{4^3(3!)^2} = \frac{720}{64 \cdot 36} = \frac{720}{2304} = \frac{5}{16}.$$

These agree term-by-term with the coefficients found above, confirming

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2}\, x^{2n}.$$

(c) (8pts) Use the alternate formulation of the series from part (b) to find the Maclaurin series for $f(x) = \sin^{-1}(x)$ for $-1 < x < 1$ (Use $\sin^{-1}(0) = 0$.)

Solution: Since $\sin^{-1}(x) = \displaystyle\int_0^x \frac{dt}{\sqrt{1-t^2}}$, integrate the series term by term:

$$\sin^{-1}(x) = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} \int_0^x t^{2n}\, dt = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)}\, x^{2n+1}.$$

The constant of integration is 0 because $\sin^{-1}(0) = 0$. Explicitly,

$$\sin^{-1}(x) = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots$$

(d) (2pts) State the radius of convergence of the series in part (c).

Solution: Term-by-term integration does not change the radius of convergence, and the binomial series $(1-x^2)^{-1/2}$ converges for $|x^2| < 1$, i.e. $|x| < 1$. Hence $R = 1$.