1. (10pts) Consider the curve $y = (2x+1)^{3/2}$ on $0 \le x \le 4$. Set up, but do NOT evaluate, an integral for the area of the surface generated by rotating this curve about the y-axis. **Fully simplify** ds.

   Solution: For $y = (2x+1)^{3/2}$, the chain rule gives

   $$y' = \frac{3}{2}(2x+1)^{1/2} \cdot 2 = 3(2x+1)^{1/2},$$

   so $(y')^2 = 9(2x+1) = 18x + 9$ and

   $$ds = \sqrt{1+(y')^2}\,dx = \sqrt{1+18x+9}\,dx = \sqrt{18x+10}\,dx.$$

   Rotating about the y-axis, the radius is $r = x$, so

   $$S = \int_0^4 2\pi\, x\, \sqrt{18x+10}\, dx.$$

2. Let $c > 0$ be a constant, and let R be the region in the first quadrant bounded above by the line $y = cx$ and below by the curve $y = x^2$. The following questions are both about this region R, but can be done independently.

   (a) (15pts) Assuming a uniform density, find the value of c for which the x-coordinate of the centroid of R is $\bar{x} = \dfrac{3}{2}$.

   Solution: The curves meet where $cx = x^2$, i.e. at $x = 0$ and $x = c$. On $[0, c]$ the line $y = cx$ is on top and $y = x^2$ is on the bottom. With constant density the ρ's cancel in the centroid, so

   $$A = \int_0^c \left(cx - x^2\right) dx = \left[\tfrac{c}{2}x^2 - \tfrac{1}{3}x^3\right]_0^c = \frac{c^3}{6},$$

   $$M_y = \int_0^c x\left(cx - x^2\right) dx = \int_0^c \left(cx^2 - x^3\right) dx = \left[\tfrac{c}{3}x^3 - \tfrac{1}{4}x^4\right]_0^c = \frac{c^4}{12}.$$

   Therefore

   $$\bar{x} = \frac{M_y}{A} = \frac{c^4/12}{c^3/6} = \frac{c}{2}.$$

   Setting $\bar{x} = \dfrac{3}{2}$ gives $\dfrac{c}{2} = \dfrac{3}{2}$, so $\boxed{c = 3}$.

   (b) (10pts) Using the value of c from part (a) (or $c = 2$ if you did not find one), set up – but do NOT evaluate – an integral using the **method of cylindrical shells** for the volume of the solid generated by rotating R about the line $x = -1$.

   Solution: With $c = 3$, the region is bounded above by $y = 3x$ and below by $y = x^2$, meeting at $x = 0$ and $x = 3$. For dx-shells about $x = -1$, the radius is $r = x + 1$ and the height is $h = 3x - x^2$. Thus

   $$V = \int_0^3 2\pi\,(x+1)\left(3x - x^2\right) dx.$$

3. (20pts) Solve the initial value problem

   $$\frac{dy}{dx} = \frac{1}{y\,(9+x^2)^{3/2}}, \qquad y(0) = 2,$$

   and express your answer in the form $y = f(x)$.

Solution: This is separable. Separate and integrate:

$$\int y\,dy = \int \frac{dx}{(9+x^2)^{3/2}}.$$

The left side gives $\dfrac{y^2}{2}$. For the right side, use the trigonometric substitution

$$x = 3\tan\theta, \qquad dx = 3\sec^2\theta\,d\theta, \qquad 9+x^2 = 9\sec^2\theta,$$

so $(9+x^2)^{3/2} = 27\sec^3\theta$. Then

$$\int \frac{dx}{(9+x^2)^{3/2}} = \int \frac{3\sec^2\theta}{27\sec^3\theta}\,d\theta = \frac{1}{9}\int \cos\theta\,d\theta = \frac{1}{9}\sin\theta + C.$$

From the reference triangle with opposite x, adjacent 3, and hypotenuse $\sqrt{x^2+9}$, we have $\sin\theta = \dfrac{x}{\sqrt{x^2+9}}$, so

$$\frac{y^2}{2} = \frac{x}{9\sqrt{x^2+9}} + C.$$

Apply $y(0) = 2$: $\dfrac{4}{2} = \dfrac{0}{9\sqrt{9}} + C \Rightarrow 2 = 0 + C \Rightarrow C = 2$. Hence

$$\frac{y^2}{2} = \frac{x}{9\sqrt{x^2+9}} + 2 \quad\Longrightarrow\quad y^2 = \frac{2x}{9\sqrt{x^2+9}} + 4.$$

Since $y(0) = 2 > 0$, take the positive root:

$$y = \sqrt{\frac{2x}{9\sqrt{x^2+9}} + 4}.$$

4. Determine whether each of the following converges or diverges. Justify your answers and name any test or theorem you use. If a quantity converges, state what it converges to.

   (a) (9pts) the sequence $a_n = \dfrac{2+\cos n}{n^2}$, for $n = 1, 2, 3, \ldots$
   Solution: Since $-1 \le \cos n \le 1$, we have $1 \le 2 + \cos n \le 3$, so for all $n \ge 1$

   $$\frac{1}{n^2} \le \frac{2+\cos n}{n^2} \le \frac{3}{n^2}.$$

   Both $\dfrac{1}{n^2} \to 0$ and $\dfrac{3}{n^2} \to 0$ as $n \to \infty$, so by the Squeeze Theorem the sequence converges, with

   $$\lim_{n\to\infty} \frac{2+\cos n}{n^2} = 0.$$

   (b) (9pts) the sequence $b_n = \dfrac{3n+2}{n+4}$, for $n = 1, 2, 3, \ldots$ You may assume the sequence is bounded. Using the Monotonic Sequence Theorem, determine whether the sequence converges.
   Solution: We are told the sequence is bounded, so by the Monotonic Sequence Theorem it suffices to show it is monotonic. Let $f(x) = \dfrac{3x+2}{x+4}$. By the quotient rule,

   $$f'(x) = \frac{3(x+4) - (3x+2)(1)}{(x+4)^2} = \frac{10}{(x+4)^2} > 0 \quad \text{for } x \ge 1,$$

so f is increasing and hence $b_n = f(n)$ is monotone increasing. (Equivalently, $b_{n+1} - b_n = \dfrac{10}{(n+5)(n+4)} > 0$.)

Since the sequence is monotone increasing and bounded, it converges by the Monotonic Sequence Theorem. It converges to

$$\lim_{n\to\infty} \frac{3n+2}{n+4} = 3.$$

(c) (9pts) the series $\displaystyle\sum_{n=1}^{\infty} \frac{5n^2+1}{2n^2+3}$

Solution: Examine the limit of the terms:

$$\lim_{n\to\infty} \frac{5n^2+1}{2n^2+3} = \lim_{n\to\infty} \frac{5+1/n^2}{2+3/n^2} = \frac{5}{2} \neq 0.$$

Since the terms do not approach 0, the series **diverges** by the Test for Divergence.

(d) (9pts) the series $\displaystyle\sum_{n=1}^{\infty} b_n$, whose nth partial sum is $s_n = \dfrac{4n}{n+1}$

Solution: By definition, the sum of a series is the limit of its partial sums (when that limit exists):

$$\sum_{n=1}^{\infty} b_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{4n}{n+1} = \lim_{n\to\infty} \frac{4}{1+1/n} = 4.$$

The limit of the partial sums exists, so the series **converges**, and its sum is 4.

(e) (9pts) the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^{2n}}$

Solution: Rewrite the terms to expose the geometric structure. Using $3^{2n} = 9^n$ and $2^{n+1} = 2 \cdot 2^n$,

$$\frac{2^{n+1}}{3^{2n}} = \frac{2 \cdot 2^n}{9^n} = 2\left(\frac{2}{9}\right)^n.$$

So

$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^{2n}} = \sum_{n=1}^{\infty} 2\left(\frac{2}{9}\right)^n.$$

This is geometric with first term (at $n=1$) $a = 2 \cdot \dfrac{2}{9} = \dfrac{4}{9}$ and ratio $r = \dfrac{2}{9}$. Since $|r| = \dfrac{2}{9} < 1$, it **converges**, with sum

$$S = \frac{a}{1-r} = \frac{4/9}{1-2/9} = \frac{4/9}{7/9} = \frac{4}{7}.$$