

1. Consider the curve given implicitly by the equation

$$3x + xy^2 = 3 + x^2y$$

- (a) (10 pts) Find  $\frac{dy}{dx}$

**Solution:**

Differentiating both sides w.r.t  $x$

$$\begin{aligned} 3 + x \left( 2y \frac{dy}{dx} \right) + y^2 &= x^2 \frac{dy}{dx} + 2xy \\ (2xy - x^2) \frac{dy}{dx} &= (2xy - y^2 - 3) \\ \frac{dy}{dx} &= \boxed{\frac{2xy - y^2 - 3}{2xy - x^2}} \end{aligned}$$

- (b) (10 pts) Find the equation of the tangent line to the curve at the point given by  $y = 0$

**Solution:**

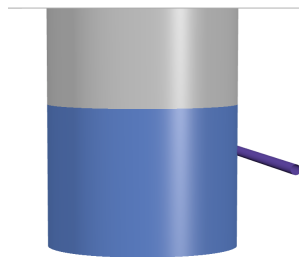
Plugging  $y = 0$  into the given equation of the curve, we get

$$\begin{aligned} 3x + 0 &= 3 + 0 \\ x &= 1 \end{aligned}$$

Plugging  $(x, y) = (1, 0)$  in the expression for derivative, we get  $\frac{dy}{dx} = \frac{-3}{-1} = 3$ .

Hence the equation of tangent line at that point is  $y - 0 = 3(x - 1)$ , or  $\boxed{y = 3x - 3}$

2. (20 pts) Water is leaking from a cylindrical tank of radius 2 m and height 10 m at the rate of  $10 \text{ m}^3/\text{s}$ . At the same time, water is being pumped into the tank at the rate of  $8 \text{ m}^3/\text{s}$ . At what rate is the height of the water level decreasing?



**Solution:**

Let the height of the water be  $h$  and the volume of the tank be  $V$ . We are given that  $\frac{dV}{dt} = -10 + 8 = -2$ . Now,

$$\begin{aligned} V &= \pi(2^2)h \\ \frac{dV}{dt} &= 4\pi \frac{dh}{dt} \\ -2 &= 4\pi \frac{dh}{dt} \\ \frac{dh}{dt} &= -\frac{2}{4\pi} \end{aligned}$$

Hence, the height of the water level decreasing at a rate of  $\boxed{-\frac{1}{2\pi} \frac{\text{m}}{\text{s}}}$

3. (20 pts) Show that  $f(x) = 2x - \sin x - 1$  has exactly one root in  $[0, \pi]$ .

(Hint: You will need the IVT as well as Rolle's theorem. For a complete solution, state the theorems clearly).

**Solution:**

The given function is a combination of a polynomial and a sine function, which means it is continuous and differentiable for all real numbers. Hence, we can first apply the Intermediate Value Theorem (IVT), which says that if  $f(x)$  changes sign in the given interval, then it has at least one root in that interval. Here

$$f(0) = -1 < 0$$

$$f(\pi) = 2\pi - 1 > 0$$

Hence there is a root in the interval  $[0, \pi]$ . Let us call this root  $x = a$

Now, let us assume there is another root,  $x = b$ . In other words,  $f(b) = 0 = f(a)$ . Hence, we can apply Rolle's Theorem, which says that there should be a turning point ( $f'(c) = 0$ ) at some point  $c$  in the interval  $(a, b)$ . Now,

$$f'(x) = 2 - \cos x$$

Hence there can't be any number  $c$  for which  $f'(c) = 0$ , because  $-1 \leq \cos x \leq 1$ . Hence, we have proven by contradiction that there can't be a second root  $x = b$ .

In conclusion  $f(x)$  has one, and only one root in the interval  $[0, \pi]$

4. Consider the function  $f(x) = \sin x - \frac{x}{\sqrt{2}}$  on the interval  $[0, 2\pi)$

- (a) (4 pts) Find the intervals where  $f(x)$  is increasing and decreasing.

**Solution:**

To find the intervals where  $f(x)$  is increasing and decreasing, we first find the critical points

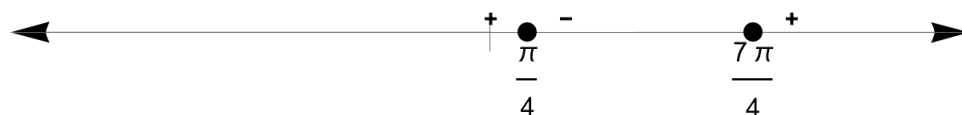
$$f'(x) = 0$$

$$\cos x - \frac{1}{\sqrt{2}} = 0$$

$$\cos x = \frac{1}{\sqrt{2}}$$

$$x = \frac{\pi}{4}, \frac{7\pi}{4}$$

Then, we perform a 1st derivative test using a sign chart around those critical points



Hence  $f(x)$  is Increasing in  $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)$  and Decreasing in  $\left(\frac{\pi}{4}, \frac{7\pi}{4}\right)$

- (b) (4 pts) Find all the local extrema

**Solution:**

From the above we conclude that, there is a local maximum at  $x = \frac{\pi}{4}$  and a local minimum at  $x = \frac{7\pi}{4}$

(c) (4 pts) Find the intervals of concavity of  $f$

**Solution:**

To find the intervals of concavity, we first find the zeroes of the 2nd derivative

$$\begin{aligned}f''(x) &= 0 \\ -\sin x &= 0 \\ x &= 0, \pi\end{aligned}$$

Since  $x = 0$  is one of the endpoints, we perform a 2nd derivative test using a sign chart around  $x = \pi$



Hence  $f(x)$  is Concave down in  $(0, \pi)$  and Concave up in  $(\pi, 2\pi)$

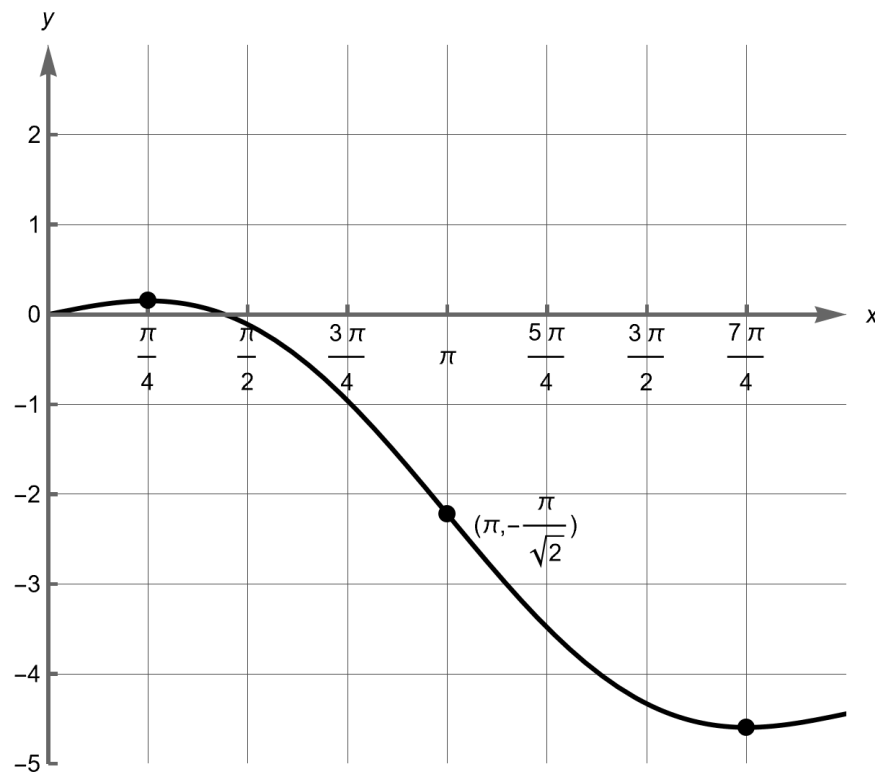
(d) (4 pts) Find the point(s) of inflection

**Solution:**

$f(x)$  has a point of inflection at  $x = \pi$

(e) (4 pts) Sketch a rough graph of the function. Make sure to label all the key points.

**Solution:**



5. (20 pts) Find the absolute maximum and minimum values of

$$f(x) = 2 \cos x - \cos 2x, \quad x \in [0, 2\pi]$$

Hint:  $\sin(2x) = 2 \sin x \cos x$

**Solution:**

First, we find the critical numbers in the given domain

$$f'(x) = 0$$

$$-2 \sin x + 2 \sin 2x = 0$$

$$-2 \sin x + 2(2 \sin x \cos x) = 0$$

$$\sin x(-1 + 2 \cos x) = 0$$

Now, the multiplicative property of zero gives us

(a)

$$\sin x = 0$$

$$x = 0, \pi$$

(b)

$$-1 + 2 \cos x = 0$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

Now, we evaluate the function at the critical numbers as well as the endpoints

$$f(0) = 1$$

$$f(\pi) = -3$$

$$f\left(\frac{\pi}{3}\right) = \frac{3}{2}$$

$$f\left(\frac{5\pi}{3}\right) = \frac{3}{2}$$

$$f(2\pi) = 1$$

Hence  $f$  has an absolute maximum value of  $\frac{3}{2}$  and an absolute minimum value of  $-3$