1. [2360/062226 (15 pts)] Consider the linear system

$$x_1 + 2x_2 + x_3 = 4$$

$$x_1 + 2x_2 = 2$$

$$x_1 + x_2 + 3x_3 = -7$$

Use the inverse of the coefficient matrix to find the solution to the system. You must use elementary row operations to find the inverse. Hint: there are no fractions in the final answer.

SOLUTION:

The system can be written as $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ with $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}$, $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{\mathbf{b}} = \begin{bmatrix} 4 \\ 2 \\ -7 \end{bmatrix}$. We need to find $\mathbf{A}^{-1}$ so that $\vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -3 & 2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 5 & 2 \\ 0 & 1 & 0 & 3 & -2 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right] \implies \mathbf{A}^{-1} = \begin{bmatrix} -6 & 5 & 2 \\ 3 & -2 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 & 5 & 2 \\ 3 & -2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -7 \end{bmatrix} = \begin{bmatrix} -28 \\ 15 \\ 2 \end{bmatrix}$$

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2. [2360/062226 (22 pts)] The following parts (a) and (b) are not related. Fully justify all of your answers.

   (a) (12 pts) Consider the set of functions in $\mathbb{P}_2$ given by $\left\{-5t+1, 4t^2-3t-2, 5t^2-\frac{13}{4}\right\}$.

      i. (6 pts) Show that the Wronskian of the functions cannot be used to determine if the functions are linearly dependent or independent on the real line.

      ii. (6 pts) Determine if the functions are linearly dependent or independent on the real line.

   (b) (10 pts) Consider the set of vectors in $\mathbb{R}^3$ given by $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$.

      i. (6 pts) Does the set of vectors span $\mathbb{R}^3$?

      ii. (4 pts) Let $\mathbf{A}$ be the matrix formed using the vectors in the set as columns. Are there vectors $\vec{\mathbf{b}} \in \mathbb{R}^3$ that are not in Col $\mathbf{A}$? Justify your answer. Hint: This can be answered without doing any computations.

SOLUTION:

(a) i.

$$W\left[-5t+1, 4t^2-3t-2, 5t^2-\tfrac{13}{4}\right](t) = \begin{vmatrix} -5t+1 & 4t^2-3t-2 & 5t^2-\frac{13}{4} \\ -5 & 8t-3 & 10t \\ 0 & 8 & 10 \end{vmatrix}$$

$$= 8(-1)^{3+2}\begin{vmatrix} -5t+1 & 5t^2-\frac{13}{4} \\ -5 & 10t \end{vmatrix} + 10(-1)^{3+3}\begin{vmatrix} -5t+1 & 4t^2-3t-2 \\ -5 & 8t-3 \end{vmatrix}$$

$$= -8\left(-50t^2+10t+25t^2-\tfrac{65}{4}\right) + 10\left(-40t^2+15t+8t-3+20t^2-15t-10\right)$$

$$= -8\left(-25t^2+10t-\tfrac{65}{4}\right) + 10\left(-20t^2+8t-13\right)$$

$$= 200t^2-80t+130-200t^2+80t-130 = 0$$

Since the Wronskian of the functions vanishes identically, we can conclude nothing about the linear dependence or independence of the functions.

ii. We can check to see if $c_1(-5t+1)+c_2\left(4t^2-3t-2\right)+c_3\left(5t^2-\frac{13}{4}\right)=0t^2+0t+0$ has nontrivial solutions. Equating coefficients on both sides yields

$$\begin{aligned} t^2 &: 4c_2+5c_3=0 \\ t^1 &: -5c_1-3c_2=0 \\ t^0 &: c_1-2c_2-\tfrac{13}{4}c_3=0 \end{aligned}$$

This can be written as

$$\begin{bmatrix} 0 & 4 & 5 \\ -5 & -3 & 0 \\ 1 & -2 & \frac{13}{4} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

with

$$\begin{vmatrix} 0 & 4 & 5 \\ -5 & -3 & 0 \\ 1 & -2 & \frac{13}{4} \end{vmatrix}=4(-1)^{1+2}\begin{vmatrix} -5 & 0 \\ 1 & \frac{13}{4} \end{vmatrix}+5(-1)^{1+3}\begin{vmatrix} -5 & -3 \\ 1 & -2 \end{vmatrix}=4\left(-\frac{65}{4}\right)+5(13)=-65+65=0$$

This means the system has nontrivial solutions implying that the functions are linearly dependent on the real line.

(b) i. There are three vectors in $\mathbb{R}^3$, a vector space of dimension 3. Check for linear independence:

$$c_1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}+c_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}+c_3\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad\Longleftrightarrow\quad \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix}=1(-1)^{1+1}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}+1(-1)^{2+1}\begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix}=-2+2=0$$

The vectors are linearly dependent and thus cannot span $\mathbb{R}^3$.

ii. Yes. Since the vectors are linearly dependent, Col $\mathbf{A}\neq\mathbb{R}^3$. Hence, there are vectors in $\mathbb{R}^3$ which cannot be written as a linear combination of the given vectors. Equivalently, there exist vectors $\vec{\mathbf{b}}\in\mathbb{R}^3$ for which $\mathbf{A}\vec{\mathbf{x}}=\vec{\mathbf{b}}$ is inconsistent. This means these vectors are not in Col $\mathbf{A}$.

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3. [2360/062226 (12 pts)] Determine if the following subsets $\mathbb{W}$ of the given vector space $\mathbb{V}$ are subspaces. Assume that the standard definitions of vector addition and scalar multiplication apply. Justify your answers.

(a) (6 pts) $\mathbb{V}=\mathbb{M}_{33}$; $\mathbb{W}$ is the set of matrices of the form $\begin{bmatrix} r & 0 & p \\ 0 & s & 0 \\ q & 0 & t \end{bmatrix}$ where p,q are real numbers and r,s,t are integers.

(b) (6 pts) $\mathbb{V}=\mathbb{R}^3$; $\mathbb{W}$ is the set of solutions to the equation $4x_1-3x_2+9x_3=0$.

SOLUTION:

(a) $\mathbb{W}$ is not a subspace. $\mathbb{W}$ is closed under vector addition (the sum of two integers is an integer) but is not closed under scalar multiplication. For example,

$$\vec{\mathbf{u}}=\mathbf{I}\in\mathbb{W} \quad\text{but}\quad \sqrt{2}\vec{\mathbf{u}}=\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}\notin\mathbb{W}$$

(b) $\mathbb{W}$ is a subspace. Use linear combinations to check for closure. Let $\alpha,\beta\in\mathbb{R}$ and

$$\vec{\mathbf{u}}=\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\in\mathbb{W}\implies 4u_1-3u_2+9u_3=0 \quad\text{and}\quad \vec{\mathbf{v}}=\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\in\mathbb{W}\implies 4v_1-3v_2+9v_3=0$$

Then $\alpha\vec{\mathbf{u}} + \beta\vec{\mathbf{v}} = \begin{bmatrix} \alpha u_1 + \beta v_1 \\ \alpha u_2 + \beta v_2 \\ \alpha u_3 + \beta v_3 \end{bmatrix}$ and

$$\begin{aligned} 4(\alpha u_1 + \beta v_1) - 3(\alpha u_2 + \beta v_2) + 9(\alpha u_3 + \beta u_3) \\ &= \alpha(4u_1 - 3u_2 + 9u_3) + \beta(4v_1 - 3v_2 + 9v_3) \\ &= \alpha(0) + \beta(0) \\ &= 0 + 0 = 0 \implies \alpha\vec{\mathbf{u}} + \beta\vec{\mathbf{v}} \in \mathbb{W} \end{aligned}$$

$\mathbb{W}$ is closed under vector addition and scalar multiplication and therefore a subspace. Note also that the equation is linear and homogeneous and hence the Superposition Principle applies, which is essentially the closure properties.

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4. [2360/062226 (18 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a two-column table (letter - answer) completely separate from any work you do to arrive at the answer.

(a) For any matrix $\mathbf{A}$, $\left|\mathbf{A}^{\mathrm{T}}\mathbf{A}\right|$ and $\left|\mathbf{A}\mathbf{A}^{\mathrm{T}}\right|$ are defined.

(b) For invertible matrices $\mathbf{A}$ and $\mathbf{B}$, $\left|\mathbf{A}\mathbf{B}^{-1}\right| = |\mathbf{A}|/|\mathbf{B}|$.

(c) If square matrix $\mathbf{A}$ has 0 for an eigenvalue, then Cramer's Rule can be used to solve the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

(d) Tr $\left[(2\mathbf{I})^3\right] = 24$ where $\mathbf{I}$ is the 3×3 identity matrix.

(e) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$

(f) Homogeneous systems of linear algebraic equations are never inconsistent.

SOLUTION:

(a) **TRUE** If $\mathbf{A}$ is $m \times n$, then $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is $n \times n$ and $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ is $m \times m$ which are both square and the determinant is defined.

(b) **TRUE** $|\mathbf{A}\mathbf{B}^{-1}| = |\mathbf{A}||\mathbf{B}^{-1}| = |\mathbf{A}||\mathbf{B}|^{-1} = |\mathbf{A}|/|\mathbf{B}|$

(c) **FALSE** If 0 is an eigenvalue of a square matrix, then $|\mathbf{A}| = 0$ and Cramer's Rule cannot be used.

(d) **TRUE** $\text{Tr}\left[(2\mathbf{I})^3\right] = \text{Tr}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}^3 = \text{Tr}\left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right) = \text{Tr}\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8 + 8 + 8 = 24$

(e) **FALSE** $c_1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ has no solution. $0c_1 + 0c_2$ can never equal 3.

(f) **TRUE** They always have at least the trivial solution.

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5. [2360/062226 (10 pts)] Consider the augmented matrix $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$ derived from the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

(a) (5 pts) If the matrix is in RREF, write the word YES. Otherwise, write NO and put the matrix in RREF.

(b) (5 pts) Find a basis for the solution space of the system. What is the dimension of the solution space?

SOLUTION:

(a) No. The following operations will put the matrix into RREF: $R_2^* = -4R_3 + R_2$, $R_2 \leftrightarrow R_3$ then $R_1^* = -1R_2 + R_1$.

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b) From the RREF, the leading/basic variables corresponding to the pivot columns are x_1 and x_4. The free variables are $x_2 = s$ and $x_3 = t$. The solutions of the system are

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \; s, t \in \mathbb{R} \quad \text{or equivalently } \vec{\mathbf{x}} \in \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

The solution space has dimension 2 with basis $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

6. [2360/062226 (23 pts)] The following parts (a) and (b) are not related.

(a) (11 pts) The characteristic polynomial of a certain matrix $\mathbf{B}$ is $p(\lambda) = 2\lambda^6 + 6\lambda^5 + 5\lambda^4$.

i. (2 pts) What is the order/size of the matrix $\mathbf{B}$?

ii. (6 pts) Find the eigenvalues of $\mathbf{B}$ and their algebraic multiplicities. Do not find the eigenvectors.

iii. (3 pts) Are the column vectors of $\mathbf{B}$ linearly dependent? Simply answer yes or no and justify your answer.

(b) (12 pts) Let $\mathbf{A} = \begin{bmatrix} -7 & 4 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 4 & -7 & -1 \\ 0 & 0 & 0 & -7 \end{bmatrix}$ and let $\mathbb{E}_{-7}$ denote the eigenspace associated with eigenvalue $\lambda = -7$.

i. (10 pts) Find a basis for $\mathbb{E}_{-7}$.

ii. (2 pts) What is the geometric multiplicity of the eigenvalue $\lambda = -7$?

SOLUTION:

(a) i. Since the characteristic polynomial is degree 6, $\mathbf{B}$ is of order 6 or 6×6.

ii.

$$2\lambda^6 + 6\lambda^5 + 5\lambda^4 = 0$$

$$\lambda^4(2\lambda^2 + 6\lambda + 5) = 0 \implies \lambda = 0 \quad \text{or} \quad 2\lambda^2 + 6\lambda + 5 = 0$$

$$\lambda = \frac{-6 \pm \sqrt{6^2 - 4(2)(5)}}{(2)(2)} = \frac{-6 \pm \sqrt{-4}}{4} = \frac{-6 \pm 2i}{4} = \frac{-3 \pm i}{2}$$

The eigenvalues of $\mathbf{B}$ are 0 with multiplicity of 4 and $\frac{-3+i}{2}, \frac{-3-i}{2}$, each with multiplicity of 1.

iii. Yes. Since $\lambda = 0$ is an eigenvalue, equivalently the column vectors of $\mathbf{B}$ are linearly dependent.

(b) i. We need to solve $(\mathbf{A} + 7\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

$$\left[\begin{array}{cccc|c} 0 & 4 & 0 & -1 & 0 \\ 0 & 8 & 0 & -2 & 0 \\ 0 & 4 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_2^* = -2R_1 + R_2 \\ R_3^* = -1R_1 + R_3 \\ R_1^* = \frac{1}{4}R_1 \end{array} \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 0 & 1 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{array}{l} v_1 = r \\ v_2 = \frac{1}{4}v_4 = \frac{1}{4}t \\ v_3 = s \\ v_4 = t \end{array}$$

A basis for $\mathbb{E}_{-7}$ is $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 4 \end{bmatrix}\right\}$

ii. Since the dimension of the eigenspace is 3, the geometric multiplicity of the eigenvalue is also 3.

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