Write your name below. This exam is worth 100 points. You must show all your work to receive full credit on the problems. You are allowed to use one page of notes (one sided). You cannot collaborate on the exam or seek outside help, nor can you use the recorded lectures, a calculator, any computational software, or material you find online.

Name:

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1. (21 points, 7 each) In each of the following problems, either show that the statement is always true or provide a counterexample that shows it is false.

   (a) If A and B are square matrices of the same size, then $A^2 - B^2 = (A+B)(A-B)$.

   (b) If A and B are skew-symmetric (or anti-symmetric) matrices of the same size, then AB is symmetric.

   (c) If A and B are square matrices of the same size and $AB = O$, then either A or B is singular.

Solution: (a) **False.** Expanding the right hand side of the equation gives

$$(A+B)(A-B) = A^2 + BA - AB - B^2 \neq A^2 - B^2$$

This only equals $A^2 - B^2$ if $BA = AB$, in other words, only when A and B commute. Any pair of non-commuting matrices works as a counterexample, e.g.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

(b) **False.** Let A and B both be antisymmetric, then

$$(AB)^T = B^T A^T = (-B)(-A) = BA \neq AB$$

One possible counterexample is

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

since

$$AB = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

which is not symmetric.

(c) **True.** Since the determinant of a sqare O matrix must be 0, we have:

$$\det AB = 0$$
$$(\det A)(\det B) = 0$$

Since determinants are real numbers, the multiplicative property of 0 tells us that either

$$\det A = 0 \text{ or } \det B = 0$$

So one of A or B must be singular.

2. (16 points) Let A $= \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 10 \\ 1 & 3 & 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}$

(a) (8 points) If A is regular find its LU factorization. If A is not regular but is also non-singular, find a permuted LU factorization.

(b) (2 points) Find the determinant of A using your factorization from part (a).

(c) (6 points) Solve the equation $A\mathbf{x} = \mathbf{b}$.

Solution: (a) We begin by making the first column upper triangular:

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 10 \\ 1 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

The matrix is not regular, so we introduce the permutation that interchanges rows 2 and 3. This makes U upper triangular, so we have the permuted LU factorization:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 10 \\ 1 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

(b) We have done one row interchange in our factorization, and since $\det U$ is the product of its diagonals, we have

$$\begin{aligned} \det A &= (-1)^1 \det U \\ &= (-1)(1)(1)(2) \\ &= -2 \end{aligned}$$

(c) Since we have a permuted LU factorization we must multiply the equation by our permutation matrix before we solve:

$$A\mathbf{x} = \mathbf{b}$$
$$PA\mathbf{x} = P\mathbf{b}$$
$$LU\mathbf{x} = P\mathbf{b}$$

First we permute $\mathbf{b}$:

$$\begin{aligned} P\mathbf{b} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 1 \\ 8 \end{pmatrix} \end{aligned}$$

Then we let $U\mathbf{x} = \mathbf{c}$ which gives $L\mathbf{c} = P\mathbf{b}$ which we solve:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 8 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

So we have $\mathbf{c} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$.

Now substitute this into $U\mathbf{x} = \mathbf{c}$ and solve:

$$\begin{aligned} \left(\begin{array}{ccr|r} 1 & 2 & 4 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 2 \end{array}\right) &\xrightarrow[R_3 \to \frac{1}{2} R_3]{} \left(\begin{array}{ccr|r} 1 & 2 & 4 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right) \\ &\xrightarrow[R_2 \to R_2 + R_3]{R_1 \to R_1 - 4R_3} \left(\begin{array}{ccc|r} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right) \\ &\xrightarrow{R_1 \to R_1 - 2R_2} \left(\begin{array}{ccc|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right) \end{aligned}$$

So our solution is

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

3. (20 points) Let $A = \begin{pmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & -1 & -3 \\ 2 & 2 & 2 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ -1 \\ 4 \end{pmatrix}$

   (a) (7 points) Find the REF of A.

   (b) (2 points) What is the rank of A?

   (c) (7 points) Find all solutions to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.

   (d) (4 points) Find the general solution to $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is the vector given above.

Solution: (a) The REF of A is found by row reduction. We're going to need to find solutions to $A\mathbf{x} = \mathbf{b}$ later, so let's make the augmented matrix $(A|\mathbf{b})$ and reduce it:

$$\left(\begin{array}{cccc|c} 2 & 1 & 3 & 4 & 5 \\ 0 & 1 & -1 & -3 & -1 \\ 2 & 2 & 2 & 1 & 4 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{} \left(\begin{array}{cccc|c} 2 & 1 & 3 & 4 & 5 \\ 0 & 1 & -1 & -3 & -1 \\ 0 & 1 & -1 & -3 & -1 \end{array}\right)$$
$$\xrightarrow[R_3 \to R_3 - R_2]{} \left(\begin{array}{cccc|c} 2 & 1 & 3 & 4 & 5 \\ 0 & 1 & -1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So the REF of our matrix is given by

$$U = \begin{pmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Since A has two pivot columns, $\text{rank}A = 2$.

(c) Using the REF of A, we see that we have two free variables, corresponding to the third and fourth columns. We therefore have two vectors we need to calculate. For the first vector we set $\mathbf{z}_3 = 1$ and $\mathbf{z}_4 = 0$ and solve $U\mathbf{z} = \mathbf{0}$:

$$\begin{pmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ 1 \\ 0 \end{pmatrix} = \mathbf{0}$$

Solving this by back substitution we have from the second row

$$\begin{aligned} v - 1 &= 0 \\ v &= 1 \end{aligned}$$

Using $v = 1$, the first row is now

$$\begin{aligned} 2u + 1 + 3 &= 0 \\ 2u &= -4 \\ u &= -2 \end{aligned}$$

So our first vector is $\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$.

For the second vector we set $\mathbf{z}_3 = 0$ and $\mathbf{z}_4 = 1$ and solve $U\mathbf{z} = \mathbf{0}$:

$$\begin{pmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ 0 \\ 1 \end{pmatrix} = \mathbf{0}$$

Solving this by back substitution we have from the second row

$$\begin{aligned} v - 3 &= 0 \\ v &= 3 \end{aligned}$$

Using $v = 3$, the first row is now

$$\begin{aligned} 2u + 3 + 4 &= 0 \\ 2u &= -7 \\ u &= -\frac{7}{2} \end{aligned}$$

So our second vector is $\begin{pmatrix} -7/2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$.

The solution to $A\mathbf{z} = \mathbf{0}$ is therefore

$$\mathbf{z} = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -7/2 \\ 3 \\ 0 \\ 1 \end{pmatrix} t$$

where s and t are any real numbers.

(d) As we have already calculated the homogeneous solution to $A\mathbf{x} = \mathbf{0}$ we need only find the particular solution and add the homogeneous solution to get the general solution. We start with the reduced augmented matrix from part (a) and continue to reduce:

$$\left(\begin{array}{cccc|c} 2 & 1 & 3 & 4 & 5 \\ 0 & 1 & -1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 \to R_1 - R_2} \left(\begin{array}{cccc|c} 2 & 0 & 4 & 7 & 6 \\ 0 & 1 & -1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We set both free variables to 0 and solve for x_1 and x_2:

$$\begin{aligned} 2x_1 &= 6 \\ x_1 &= 3 \\ x_2 &= -1 \end{aligned}$$

So after adding the homogeneous solution we find the general solution to be

$$\mathbf{x} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -7/2 \\ 3 \\ 0 \\ 1 \end{pmatrix} t$$

4. (12 points) Let $A = \begin{pmatrix} k & 1 \\ 1 & 2k \end{pmatrix}$ with $k \in \mathbb{R}$ and let $\mathbf{b} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

   (a) (6 points) For what values of k is the system $A\mathbf{x} = \mathbf{b}$ guaranteed to have a unique solution?

   (b) (6 points) For each k value where $A\mathbf{x} = \mathbf{b}$ doesn't have a unique solution, what is the compatibility condition on $\mathbf{b}$?

Solution: (a) We make the augmented matrix and reduce:

$$\left(\begin{array}{cc|c} k & 1 & x \\ 1 & 2k & y \end{array}\right) \xrightarrow[R_2 \to R_2 - \frac{1}{k}R_1]{} \left(\begin{array}{cc|c} k & 1 & x \\ 0 & 2k - \frac{1}{k} & y - \frac{x}{k} \end{array}\right)$$

To have a unique solution, we must have a non-zero entry in the bottom right of the REF matrix, so we have

$$\begin{aligned} 2k - \frac{1}{k} &\neq 0 \\ 2k^2 - 1 &\neq 0 \\ k^2 &\neq 1 \\ k &\neq \pm\frac{1}{\sqrt{2}} \end{aligned}$$

So for any real value for k except these two the system will have unique solutions.

(b) The system won't have unique solutions when the bottom right entry in the reduced matrix equals 0. This happens when k equals one of the two values from part (a). For our compatibility conditions, we set the last entry in the vector equal to zero for the k value:

$k = 1/\sqrt{2}$:

$$\begin{aligned} y - \sqrt{2}x &= 0 \\ y &= \sqrt{2}x \end{aligned}$$

$k = -1/\sqrt{2}$:

$$\begin{aligned} y + \sqrt{2}x &= 0 \\ y &= -\sqrt{2}x \end{aligned}$$

5. (15 points) Let A be a square matrix which satisfies the equation $A = A^3$.

   (a) (8 points) What are the possible values for the determinant of A?

   (b) (2 points) Will A always be invertible?

   (c) (5 points) Assuming A is invertible, find an expression for A^{-1} in terms of A.

Solution: (a) Using the rules for determinants we see that, starting with $A = A^3$:

$$\begin{aligned} A &= A^3 \\ \det A &= \det A^3 \\ \det A &= (\det A)^3 \\ 0 &= (\det A)^3 - \det A \\ 0 &= ((\det A)^2 - 1)\det A \\ 0 &= (\det A - 1)(\det A + 1)\det A \end{aligned}$$

So the possible values are 1, 0, and -1.

(b) Since 0 is one of our possible determinants, A may be singular and hence A isn't necessarily invetible.

(c) If $\det A \neq 0$ then A^{-1} exists. Again starting with $A = A^3$ we have

$$\begin{aligned} A &= A^3 \\ A^{-1}A &= A^{-1}A^3 \\ I &= A^2 \end{aligned}$$

Since $AA = I$, we see that A is it's own inverse:

$$A^{-1} = A$$

6. (12 points) Consider $\mathbb{R}^{2\times 2}$, the vector space of 2×2 real matrices. Let $\mathbb{S}$ be the subset of these matrices given by

$$\mathbb{S} = \left\{ \begin{pmatrix} 0 & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$$

Is $\mathbb{S}$ a vector subspace of $\mathbb{R}^{2\times 2}$? Either prove that it is or show that it is not.

Solution: Yes, this is a vector subspace. It is not empty, as $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{S}$. (Set $a = b = 0$.)

It is closed on scalar multiplication as

$$c\begin{pmatrix} 0 & b \\ -b & a \end{pmatrix} = \begin{pmatrix} 0 & (cb) \\ -(cb) & (ca) \end{pmatrix} \in \mathbb{S}$$

and vector additions as

$$\begin{pmatrix} 0 & b \\ -b & a \end{pmatrix} + \begin{pmatrix} 0 & c \\ -c & d \end{pmatrix} = \begin{pmatrix} 0 & (b+c) \\ -(b+c) & (a+d) \end{pmatrix} \in \mathbb{S}$$