

Write your name below. This exam is worth 100 points and has 6 problems. You must show your work to receive full credit on the problems. You are allowed to use one side of an 8.5x11 page of notes. You cannot collaborate on the exam or seek outside help, nor can you use the recorded lectures, a calculator, any computational software, or material you find online.

Name:

1. (24 points, 8 each) In each of the following problems, either show that the statement is always true or provide a counterexample that shows it is false.

- (a) If A and B are square matrices of the same size, then $A^2 - B^2 = (A + B)(A - B)$.
- (b) If A and B are skew-symmetric (or anti-symmetric) matrices of the same size, then AB is symmetric.
- (c) If A and B are square matrices of the same size and $AB = O$, then either A or B is singular.

2. (16 points) Let $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 10 \\ 1 & 3 & 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}$

- (a) (8 points) If A is regular find its LU factorization. If A is not regular but is also non-singular, find a permuted LU factorization.
- (b) (2 points) Find the determinant of A using your factorization from part (a).
- (c) (6 points) Solve the equation $A\mathbf{x} = \mathbf{b}$.

3. (20 points) Let $A = \begin{pmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & -1 & -3 \\ 2 & 2 & 2 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ -1 \\ 4 \end{pmatrix}$

- (a) (7 points) Find the REF of A .
- (b) (2 points) What is the rank of A ?
- (c) (7 points) Find all solutions to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.
- (d) (4 points) Find the general solution to $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is the vector given above.

4. (12 points) Let $A = \begin{pmatrix} k & 1 \\ 1 & 2k \end{pmatrix}$ with $k \in \mathbb{R}$ and let $\mathbf{b} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

- (a) (6 points) For what values of k is the system $A\mathbf{x} = \mathbf{b}$ guaranteed to have a unique solution?
- (b) (6 points) For each k value where $A\mathbf{x} = \mathbf{b}$ doesn't have a unique solution, what is the compatibility condition on $\mathbf{b}$?

5. (16 points) Let A be a square matrix which satisfies the equation $A = A^3$.
- (a) (8 points) What are the possible values for the determinant of A ?
 - (b) (2 points) Will A always be invertible?
 - (c) (6 points) Assuming A is invertible, find an expression for A^{-1} in terms of A .

6. (12 points) Consider $\mathbb{R}^{2 \times 2}$, the vector space of 2×2 real matrices. Let $\mathbb{S}$ be the subset of these matrices given by

$$\mathbb{S} = \left\{ \begin{pmatrix} 0 & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$$

Is $\mathbb{S}$ a vector subspace of $\mathbb{R}^{2 \times 2}$? Either prove that it is or show that it is not.