1. (24 pts) If the statement is **always true**, write "TRUE"; if it is possible for the statement to be false then mark "FALSE." You must give a **justification** for your answer. That is, if the answer is true, provide a brief proof. If the answer is false, provide a counterexample.

   (a) Let L be a linear transformation, $L : \mathbb{R}^2 \to \mathbb{R}^2$. L will always send a straight line in $\mathbb{R}^2$ to either a line or a single point in $\mathbb{R}^2$.

   (b) If the square matrix A is complete, then A^2 is also complete.

   (c) Let A be a real $m \times n$ matrix. Then $\ker(A^T A) = \ker(A)$.

   (d) If A is a real 2×2 matrix and $\mathbf{x}^T A\,\mathbf{x} > 0$ for every nonzero $\mathbf{x} \in \mathbb{R}^2$, then A is symmetric.

Solution

(a) **TRUE.** Any (affine) line in $\mathbb{R}^2$ can be written as

$$\ell = \{\mathbf{p} + t\mathbf{v} : t \in \mathbb{R}\}$$

for some fixed $\mathbf{p}, \mathbf{v} \in \mathbb{R}^2$ with $\mathbf{v} \neq \mathbf{0}$. Since L is linear,

$$L(\ell) = \{L(\mathbf{p} + t\mathbf{v}) : t \in \mathbb{R}\} = \{L(\mathbf{p}) + t\,L(\mathbf{v}) : t \in \mathbb{R}\}.$$

If $L(\mathbf{v}) = \mathbf{0}$ then $L(\ell) = \{L(\mathbf{p})\}$ is a single point; otherwise $L(\ell)$ is a line with direction vector $L(\mathbf{v})$.

(b) **TRUE.** "Complete" means A has a basis of eigenvectors (equivalently, A is diagonalizable). Thus there is an invertible S and a diagonal D with $A = SDS^{-1}$. Squaring gives

$$A^2 = (SDS^{-1})(SDS^{-1}) = SD^2S^{-1},$$

and D^2 is diagonal. Hence A^2 is also diagonalizable (so complete).

(c) **TRUE.** If $A\mathbf{x} = \mathbf{0}$, then $A^T A\,\mathbf{x} = A^T\mathbf{0} = \mathbf{0}$, so $\ker(A) \subseteq \ker(A^T A)$. Conversely, if $A^T A\,\mathbf{x} = \mathbf{0}$, then

$$0 = \mathbf{x}^T A^T A\,\mathbf{x} = (A\mathbf{x})^T(A\mathbf{x}) = \|A\mathbf{x}\|^2,$$

forcing $A\mathbf{x} = \mathbf{0}$, so $\ker(A^T A) \subseteq \ker(A)$. Hence the two kernels coincide.

(d) **FALSE.** For any square matrix A, $\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T\big(\frac{1}{2}(A + A^T)\big)\mathbf{x}$, so only the symmetric part of A enters the quadratic form. Take

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then $\mathbf{x}^T A\,\mathbf{x} = x_1^2 + x_2^2 > 0$ for all $\mathbf{x} \neq \mathbf{0}$, but A is not symmetric.

2. (25 pts) Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$. One of these matrices is diagonalizable and the other is not.

   (a) (15 pts) For the diagonalizable matrix, find a matrix S that diagonalizes it.

   (b) (10 pts) For the matrix that cannot be diagonalized, what is its Jordan canonical form?

Solution

(a) Both A and B are upper triangular, so their eigenvalues are the diagonal entries: 1 (with algebraic multiplicity 2) and 2 (with algebraic multiplicity 1).

For A,

$$A - I = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The equations $(A - I)\mathbf{x} = \mathbf{0}$ give $x_2 = 0$, with x_1, x_3 free, so $\dim E_{\lambda=1}(A) = 2$. Since this equals the algebraic multiplicity of $\lambda = 1$, the matrix A is diagonalizable.

A basis of eigenvectors is

$$\mathbf{s}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{s}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad (\lambda = 1),$$

and for $\lambda = 2$ we solve

$$A - 2I = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which gives $x_1 = 2x_2$ and $x_3 = 0$, so we may take

$$\mathbf{s}_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \qquad (\lambda = 2).$$

Let $S = [\mathbf{s}_1 \ \mathbf{s}_2 \ \mathbf{s}_3]$ and $D = \mathrm{diag}(1, 1, 2)$. Then $A = SDS^{-1}$.

(b) For B,

$$B - I = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The equation $(B - I)\mathbf{x} = \mathbf{0}$ gives $x_2 = 0$ from the first row, and then $x_3 = 0$ from the second row, so $\dim E_{\lambda=1}(B) = 1 < 2$. Hence B is not diagonalizable.

Since $\lambda = 1$ has algebraic multiplicity 2 and geometric multiplicity 1, the Jordan canonical form of B has a single Jordan block of size 2 for $\lambda = 1$ and a 1×1 block for $\lambda = 2$:

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

3. (29 pts) Let A be the matrix with the singular value decomposition

$$A = \begin{pmatrix} 2/3 & -1/\sqrt{2} \\ 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 3\sqrt{13} & 0 \\ 0 & \sqrt{26} \end{pmatrix} \begin{pmatrix} 2/\sqrt{13} & 3/\sqrt{13} \\ -3/\sqrt{13} & 2/\sqrt{13} \end{pmatrix}$$

(a) (3 pts) What is the rank of A?

(b) (10 pts) Find A's best rank 1 approximation, A_1.

(c) (16 pts) Find the pseudoinverse of A_1. (That is, the psuedoinverse of the approximation, not A itself.)

Solution:

(a) A has two singular values, so the rank of A is 2.

(b)

$$\begin{aligned} A_1 &= \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} 3\sqrt{13}\frac{1}{\sqrt{13}} \begin{pmatrix} 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 6 \\ 2 & 3 \\ 4 & 6 \end{pmatrix} \end{aligned}$$

(c)

$$\begin{aligned} A_1^+ &= \frac{1}{\sqrt{13}}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \frac{1}{3\sqrt{13}}\frac{1}{3} \begin{pmatrix} 2 & 1 & 2 \end{pmatrix} \\ &= \frac{1}{9}\cdot\frac{1}{13}\begin{pmatrix} 4 & 2 & 4 \\ 6 & 3 & 6 \end{pmatrix} \end{aligned}$$

4. (20 pts) Let $p(\mathbf{x}) = 5x^2 + 2xy - 4xz + y^2 + z^2 - 2x + 2y + 2z$.

   (a) (7 pts) Show that the quadratic function $p(\mathbf{x})$ has infinitely many minimizers.

   (b) (7 pts) Find an expression for all of the minimizers.

   (c) (6 pts) What is the minimum value of $p(\mathbf{x})$?

Solution:

(a) We rcognize that

$$p(\mathbf{x}) = \mathbf{x}^T K \mathbf{x} - 2\mathbf{x}^T \mathbf{f}$$

where

$$K = \begin{pmatrix} 5 & 1 & -2 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

Solving $K\mathbf{x} = \mathbf{f}$:

$$\left(\begin{array}{ccc|c} 5 & 1 & -2 & 1 \\ 1 & 1 & 0 & -1 \\ -2 & 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 5 & 1 & -2 & 1 \\ 0 & 4/5 & 2/5 & -6/5 \\ 0 & 2/5 & 1/5 & -3/5 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 5 & 1 & -2 & 1 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since the system is consistent and K is positive semidefinite, there are infinitely many minimizers.

(b) The minimizers are located at the solutions:

$$\left(\begin{array}{ccc|c} 5 & 1 & -2 & 1 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 5 & 0 & -5/2 & 5/2 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 1/2 & -3/2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\mathbf{v} = \begin{pmatrix} 1/2 \\ -3/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} t$$

(c) To find the minimum value, we substitute one of the minimizers into $p(\mathbf{x})$:

$$
\begin{aligned}
p\begin{pmatrix} 1/2 \\ -3/2 \\ 0 \end{pmatrix} &= 5(1/2)^2 + 2(1/2)(-3/2) - 4(1/2)(0) + (-3/2)^2 + 0^2 - 2(1/2) + 2(-3/2) + 2(0 \\
&= 5/4 - 3/2 - 2 + 9/4 - 1 - 3 \\
&= -2
\end{aligned}
$$

Alternatively, we could use one of the formulas involving K and $\mathbf{f}$ and $c(= 0)$:

$$
-\mathbf{x}^{*T}\mathbf{f} = -\begin{pmatrix} \frac{1}{2} & -\frac{3}{2} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = -2
$$

5. (24 pts) Consider the inner product

$$\langle f,\, g\rangle \;=\; \int_{-1}^{1} f(x)\, g(x)\, dx$$

on the space of continuous functions on $[-1,1]$. Let $f(x) = x^3$, and let $\mathcal{P}_1 = \mathrm{span}\{1,\, x\}$ denote the space of polynomials of degree at most 1.

(a) (12 pts) Find the best approximation $p^\star(x) = c_0 + c_1 x$ to $f(x) = x^3$ in $\mathcal{P}_1$.

(b) (12 pts) Verify that the residual $r(x) = f(x) - p^\star(x)$ is orthogonal to $\mathcal{P}_1$ by computing $\langle r,\, 1\rangle$ and $\langle r,\, x\rangle$ directly.

Solution:

(a) We project the function x^3 onto the functions 1 and x using the inner product above to find the best approximation. Our basis is orthogonal since

$$\langle 1, x\rangle = \int_{-1}^{1} x dx = 0$$

so we can use our projection formulas to find

$$c_0 = \frac{\langle 1, x^3\rangle}{\langle 1, 1\rangle} = \frac{\int_{-1}^{1} x^3 dx}{\int_{-1}^{1} 1 dx} = 0$$

$$c_1 = \frac{\langle x, x^3\rangle}{\langle x, x\rangle} = \frac{\int_{-1}^{1} x^4 dx}{\int_{-1}^{1} x^2 dx} = \frac{2/5}{2/3} = \frac{3}{5}$$

Therefore

$$p^\star(x) = \tfrac{3}{5}\, x.$$

(b) The residual is $r(x) = x^3 - \frac{3}{5}\, x$. Compute:

$$\langle r, 1\rangle = \int_{-1}^{1} \big(x^3 - \tfrac{3}{5}x\big)\, dx = 0 \;\; \text{(odd integrand).} \checkmark$$

$$\langle r, x\rangle = \int_{-1}^{1} \big(x^4 - \tfrac{3}{5}x^2\big)\, dx = \tfrac{2}{5} - \tfrac{3}{5} \cdot \tfrac{2}{3} = \tfrac{2}{5} - \tfrac{2}{5} = 0. \checkmark$$

Since r is orthogonal to both 1 and x, it is orthogonal to every element of $\mathcal{P}_1 = \mathrm{span}\{1, x\}$. This confirms that $p^\star$ is the orthogonal projection of f onto $\mathcal{P}_1$, hence the best $\mathcal{P}_1$-approximation to f in the $L^2[-1,1]$ sense.

6. (14 pts) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. Find matrices T and S such that $B = T^{-1}AS$ is in canonical form.

Solution:

A is rank 2:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \end{pmatrix}$$

so the first two columns of S will be the transposed rows of A as these form a basis for the coimage. The last column of S must be a basis vector for the kernel:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\mathbf{z} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

So we have

$$S = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & -1 \end{pmatrix}$$

To find T we calculate A times the first two columns of S:

$$\begin{aligned} T &= \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix} \end{aligned}$$

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7. (14 pts) Let A be a symmetric 2×2 matrix with eigenvalues 1 and 2.

   (a) (7 pts) If the eigenvector for $\lambda = 1$ is $\mathbf{s}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, what is the eigenvector for $\lambda = 2$?

   (b) (7 pts) Find e^{At}

Solution:

(a) Since A is symmetric, eigenvectors for distinct eigenvalues are orthogonal. Thus an eigenvector for $\lambda = 2$ must be orthogonal to $\mathbf{s}_1 = (1, -2)^T$. One choice is

$$\mathbf{s}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(b) Let $\mathbf{u}_1 = \frac{1}{\sqrt{5}}(1, -2)^T$ and $\mathbf{u}_2 = \frac{1}{\sqrt{5}}(2, 1)^T$, and set $U = [\mathbf{u}_1 \ \mathbf{u}_2]$. Then

$$A = U \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} U^T,$$

so

$$e^{At} = U \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} U^T = \frac{1}{5} \begin{pmatrix} e^t + 4e^{2t} & -2e^t + 2e^{2t} \\ -2e^t + 2e^{2t} & 4e^t + e^{2t} \end{pmatrix}.$$