1. (24 pts) If the statement is **always true**, write "TRUE"; if it is possible for the statement to be false then mark "FALSE." You must give a **justification** for your answer. That is, if the answer is true, provide a brief proof. If the answer is false, provide a counterexample.

   (a) Let L be a linear transformation, $L : \mathbb{R}^2 \to \mathbb{R}^2$. L will always send a straight line in $\mathbb{R}^2$ to either a line or a single point in $\mathbb{R}^2$.

   (b) If the square matrix A is complete, then A^2 is also complete.

   (c) Let A be a real $m \times n$ matrix. Then $\ker(A^T A) = \ker(A)$.

   (d) If A is a real 2×2 matrix and $\mathbf{x}^T A\,\mathbf{x} > 0$ for every nonzero $\mathbf{x} \in \mathbb{R}^2$, then A is symmetric.

2. (25 pts) Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$. One of these matrices is diagonalizable and the other is not.

   (a) (15 pts) For the diagonalizable matrix, find a matrix S that diagonalizes it.

   (b) (10 pts) For the matrix that cannot be diagonalized, what is its Jordan canonical form?

3. (29 pts) Let A be the matrix with the singular value decomposition

$$A = \begin{pmatrix} 2/3 & -1/\sqrt{2} \\ 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 3\sqrt{13} & 0 \\ 0 & \sqrt{26} \end{pmatrix} \begin{pmatrix} 2/\sqrt{13} & 3/\sqrt{13} \\ -3/\sqrt{13} & 2/\sqrt{13} \end{pmatrix}$$

   (a) (3 pts) What is the rank of A?

   (b) (10 pts) Find A's best rank 1 approximation, A_1.

   (c) (16 pts) Find the pseudoinverse of A_1. (That is, the psuedoinverse of the approximation, not A itself.)

4. (20 pts) Let $p(\mathbf{x}) = 5x^2 + 2xy - 4xz + y^2 + z^2 - 2x + 2y + 2z$.

   (a) (7 pts) Show that the quadratic function $p(\mathbf{x})$ has infinitely many minimizers.

   (b) (7 pts) Find an expression for all of the minimizers.

   (c) (6 pts) What is the minimum value of $p(\mathbf{x})$?

5. (24 pts) Consider the inner product

$$\langle f,\, g\rangle \;=\; \int_{-1}^{1} f(x)\, g(x)\, dx$$

on the space of continuous functions on $[-1, 1]$. Let $f(x) = x^3$, and let $\mathcal{P}_1 = \operatorname{span}\{1,\, x\}$ denote the space of polynomials of degree at most 1.

(a) (12 pts) Find the best approximation $p^\star(x) = c_0 + c_1 x$ to $f(x) = x^3$ in $\mathcal{P}_1$.

(b) (12 pts) Verify that the residual $r(x) = f(x) - p^\star(x)$ is orthogonal to $\mathcal{P}_1$ by computing $\langle r,\, 1\rangle$ and $\langle r,\, x\rangle$ directly.

6. (14 pts) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. Find matrices T and S such that $B = T^{-1}AS$ is in canonical form.

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7. (14 pts) Let A be a symmetric 2×2 matrix with eigenvalues 1 and 2.

   (a) (7 pts) If the eigenvector for $\lambda = 1$ is $\mathbf{s}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, what is the eigenvector for $\lambda = 2$?

   (b) (7 pts) Find e^{At}