Write your name below and circle and your professor's name and section number below. You are *not* allowed to use textbooks, notes, or a calculator. To receive full credit on a problem you must show **sufficient justification for your conclusion** unless explicitly stated otherwise.

Name:

Instructor and Section: 001 - Bortz 002 - Mitchell 003 - Beylkin

1. (24 pts) If the statement is **always true**, write "TRUE"; if it is possible for the statement to be false then mark "FALSE." You must give a **justification** for your answer. That is, if the answer is true, provide a brief proof. If the answer is false, provide a counterexample.

   (a) Every norm on $\mathbb{R}^n$ is induced by an inner product.

   (b) The set of complex vectors of the form $\begin{pmatrix} z \\ \overline{z} \end{pmatrix}$, where $z \in \mathbb{C}$, forms a subspace of the complex vector space $\mathbb{C}^2$.

   (c) The vector product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\cos(x)dx$ is a valid inner product on C^0.

   (d) If A is an $n \times n$ matrix, then $\ker A$ and $\operatorname{img} A$ are orthogonal subspaces of $\mathbb{R}^n$

Solution

(a) False. The 1-norm $\|\mathbf{x}\|_1 = |x_1| + \cdots + |x_n|$ is a valid norm but does not satisfy the parallelogram law:

$$\|\mathbf{u}+\mathbf{v}\|_1^2 + \|\mathbf{u}-\mathbf{v}\|_1^2 \neq 2\|\mathbf{u}\|_1^2 + 2\|\mathbf{v}\|_1^2$$

in general. For example, with $\mathbf{u} = (1,0)^T$ and $\mathbf{v} = (0,1)^T$, the left side gives $4 + 4 = 8$ while the right gives $2 + 2 = 4$. Since any norm induced by an inner product must satisfy the parallelogram law, $\|\cdot\|_1$ is not induced by any inner product.

(b) False. Not closed under scalar multiplication

(c) True. As an integral, it is symmetric and bilinear. Positivity also holds since $\cos(x) > 0$ on the interval. So it is a valid inner product.

(d) False. Here is a counterexample:

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

$$\ker A = \text{Span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

$$\operatorname{img} A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

Since

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \neq 0$$

they are not orthogonal subspaces of $\mathbb{R}^2$

2. (20 pts) Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

(a) (5 pts) Which of these matrices can be used to define an inner product on $\mathbb{R}^3$?

(b) (5 pts) Do either of these matrices have null directions? If so find them.

(c) (5 pts) Find the quadratic form using B.

(d) (5 pts) Factor the quadratic form from part (c).

Solution

(a) A is positive semidefinite and so cannot define an inner product:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

B is positive definite and defines an inner product:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 0 & 3/5 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 0 & 5/3 & 1 \\ 0 & 0 & 2/5 \end{pmatrix}$$

(b) Since A is positive semidefinite, it will have at least one null direction. We find them by calculating a basis for $\ker A$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{z} = \mathbf{0}$$

$$\mathbf{z} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

(c) Reading off the coefficients, we find

$$q(\mathbf{x}) = 3x_1^2 + 2x_1x_2 + 2x_2^2 + 2x_2x_3 + x_3^2$$

(d) We use L^T to perform our change of variables:

$$\mathbf{y} = \begin{pmatrix} 1 & 1/3 & 0 \\ 0 & 1 & 3/5 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2/3 \\ x_2 + 3x_3/5 \\ x_3 \end{pmatrix}$$

Then the quadratic form becomes $\mathbf{y}^T D\mathbf{y}$:

$$q(\mathbf{x}) = 3(x_1 + \frac{1}{3}x_2)^2 + \frac{5}{3}(x_2 + \frac{3}{5}x_3)^2 + \frac{2}{5}x_3^2$$

3. (20 pts) Consider matrix

$$A = \begin{pmatrix} 1 & 5 & 0 & -3 \\ 2 & 10 & 1 & -4 \\ -1 & -5 & 1 & 5 \end{pmatrix}$$

   (a) Find bases for the image and co-image of A
   (b) Find basis for the null space (kernel) of A
   (c) Find basis for the co-kernel of A by using its orthogonality to the image

Solution:

We first find the REF of A:

$$\begin{pmatrix} 1 & 5 & 0 & -3 \\ 2 & 10 & 1 & -4 \\ -1 & -5 & 1 & 5 \end{pmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 5 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\xrightarrow[R_3 \to R_3 - R_2]{} \begin{pmatrix} 1 & 5 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) The first and third columns of A are the pivot columns, so we take them as our basis for the image:

$$\text{img} A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

For the coimage, we take the transposes of the nonzero rows of the REF as the basis:

$$\text{coimg} A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 5 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

(b) For the kernel's basis vector, we solve the homogeneous equation $A\mathbf{x} = \mathbf{0}$:

$$\ker A = \text{Span}\left\{ \begin{pmatrix} -5 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$$

(c) The cokernel's basis vector must be orthogonal to both of the image basis vectors.

Let $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 1 \end{pmatrix}$ Then we have

$$B^T \mathbf{z} = \mathbf{0}$$

for the members of the cokernel. Solving this equation gives

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{z} = \mathbf{0}$$

$$\text{coker} A = \text{Span} \left\{ \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\}$$

4. (20 pts) Let

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 2 & 0 & 1 \\ -1 & 1 & 4 \end{pmatrix}$$

(a) (14 pts) Find the QR factorization of A using the Gram-Schmidt procedure.

(b) (6 pts) If you used Householder matrices instead to factor A into QR, you would find the first matrix product to be

$$H_1 A = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & 3 \end{pmatrix}$$

What is the **second** Householder matrix needed to factor A?

Solution:

(a) We normalize the first column to get the first column in Q:

$$\mathbf{q}_1 = \frac{\mathbf{a}_1}{|\mathbf{a}_1|} = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

Next we calculate r_{12} and subtract the component along $\mathbf{q}_1$ from $\mathbf{a}_1$ and normalize:

$$r_{12} = \mathbf{q}_1^T \mathbf{a}_2 = \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = -1$$

$$\mathbf{v}_2 = \mathbf{a}_2 - r_{12}\mathbf{q}_1$$

$$\mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - (-1)\frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

Since $||\mathbf{v}_2|| = 1$ we have $\mathbf{q}_2 = \mathbf{v}_2$ and $r_{22} = 1$.

$$\mathbf{q}_2 = \frac{1}{3}\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

Finally, we find r_{13} and r_{23}, subtract off the components along $\mathbf{q}_1$ and $\mathbf{q}_2$ from $\mathbf{a}_2$ and normalize.

$$r_{13} = \mathbf{q}_1^T \mathbf{a}_3 = \frac{1}{2} \begin{pmatrix} 2 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = 0$$

$$r_{23} = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = 3$$

$$\mathbf{v}_3 = \mathbf{a}_3 - r_{13}\mathbf{q}_1 - r_{23}\mathbf{q}_2$$

$$= \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} - 3\frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

Normalizing gives

$$r_{33} = ||\mathbf{v}_3|| = 3$$

$$\mathbf{q}_3 = \frac{1}{3} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

Bringing this all together we get

$$Q = \frac{1}{3} \begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix} \qquad R = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$

(b) We need the H matrix that sends $\mathbf{e}_3$ to $\mathbf{e}_2$. This is just the permutation matrix

$$H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

5. (16 pts)

   (a) (6 pts) Determine which of the following are valid norms on $\mathbb{R}^2$. Justify your answer by verifying the norm axioms or identifying which axiom fails.

   (i) $\|\mathbf{x}\| = \sqrt{x_1^2 + 4x_2^2}$

   (ii) $\|\mathbf{x}\| = |x_1 - x_2|$

   (b) (5 pts) Using the inner product $\langle \mathbf{x}, \mathbf{y} \rangle_B = \mathbf{x}^T B \mathbf{y}$ with

$$B = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

   compute $\|\mathbf{e}_1\|_B$, $\|\mathbf{e}_2\|_B$, and $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle_B$.

   (c) (5 pts) Verify the Cauchy–Schwarz inequality for $\mathbf{e}_1$ and $\mathbf{e}_2$ with respect to $\langle \cdot, \cdot \rangle_B$.

Solution:

(a) (i) This **is** a valid norm. It is the norm induced by the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + 4x_2 y_2$ (equivalently, $\mathbf{x}^T \mathrm{diag}(1,4)\, \mathbf{y}$). Since $\mathrm{diag}(1,4)$ is positive definite, the induced norm satisfies all norm axioms.

Alternatively, one can verify directly:

- **Positive definiteness:** $\sqrt{x_1^2 + 4x_2^2} = 0$ iff $x_1 = x_2 = 0$.
- **Homogeneity:** $\|c\mathbf{x}\| = \sqrt{c^2 x_1^2 + 4c^2 x_2^2} = |c|\|\mathbf{x}\|$.
- **Triangle inequality:** Follows from Minkowski's inequality (or from being an inner product norm).

(ii) This is **not** a valid norm. Positive definiteness fails: take $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then $\|\mathbf{x}\| = |1 - 1| = 0$, but $\mathbf{x} \neq \mathbf{0}$.

(b) We have $B = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Then:

$$\begin{aligned} \|\mathbf{e}_1\|_B &= \sqrt{\mathbf{e}_1^T B \mathbf{e}_1} = \sqrt{3} \\ \|\mathbf{e}_2\|_B &= \sqrt{\mathbf{e}_2^T B \mathbf{e}_2} = \sqrt{2} \\ \langle \mathbf{e}_1, \mathbf{e}_2 \rangle_B &= \mathbf{e}_1^T B \mathbf{e}_2 = 1 \end{aligned}$$

(c) The Cauchy–Schwarz inequality states $|\langle \mathbf{e}_1, \mathbf{e}_2 \rangle_B| \leq \|\mathbf{e}_1\|_B \, \|\mathbf{e}_2\|_B$. Substituting:

$$\begin{aligned} |1| &\leq \sqrt{3} \cdot \sqrt{2} = \sqrt{6} \\ 1 &\leq \sqrt{6} \approx 2.449 \quad \checkmark \end{aligned}$$

The inequality holds.