Write your name below and circle and your professor's name and section number below. You are *not* allowed to use textbooks, notes, or a calculator. To receive full credit on a problem you must show **sufficient justification for your conclusion** unless explicitly stated otherwise.

Name:

Instructor and Section: 001 - Bortz 002 - Mitchell 003 - Beylkin

1. (24 pts) If the statement is **always true**, write "TRUE"; if it is possible for the statement to be false then mark "FALSE." You must give a **justification** for your answer. That is, if the answer is true, provide a brief proof. If the answer is false, provide a counterexample.

   (a) Every norm on $\mathbb{R}^n$ is induced by an inner product.

   (b) The set of complex vectors of the form $\begin{pmatrix} z \\ \bar{z} \end{pmatrix}$, where $z \in \mathbb{C}$, forms a subspace of the complex vector space $\mathbb{C}^2$.

   (c) The vector product $\langle f, g \rangle = \displaystyle\int_{-1}^{1} f(x)g(x)\cos(x)dx$ is a valid inner product on C^0.

   (d) If A is an $n \times n$ matrix, then $\ker A$ and $\mathrm{img} A$ are orthogonal subspaces of $\mathbb{R}^n$

2. (20 pts) Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

   (a) (5 pts) Which of these matrices can be used to define an inner product on $\mathbb{R}^3$?

   (b) (5 pts) Do either of these matrices have null directions? If so find them.

   (c) (5 pts) Find the quadratic form using B.

   (d) (5 pts) Factor the quadratic form from part (c).

3. (20 pts) Consider matrix

$$A = \begin{pmatrix} 1 & 5 & 0 & -3 \\ 2 & 10 & 1 & -4 \\ -1 & -5 & 1 & 5 \end{pmatrix}$$

(a) Find bases for the image and co-image of A

(b) Find basis for the null space (kernel) of A

(c) Find basis for the co-kernel of A by using its orthogonality to the image

4. (20 pts) Let

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 2 & 0 & 1 \\ -1 & 1 & 4 \end{pmatrix}$$

(a) (14 pts) Find the QR factorization of A using the Gram-Schmidt procedure.

(b) (6 pts) If you used Householder matrices instead to factor A into QR, you would find the first matrix product to be

$$H_1 A = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & 3 \end{pmatrix}$$

What is the **second** Householder matrix needed to factor A?

5. (16 pts)

   (a) (6 pts) Determine which of the following are valid norms on $\mathbb{R}^2$. Justify your answer by verifying the norm axioms or identifying which axiom fails.

      (i) $\|\mathbf{x}\| = \sqrt{x_1^2 + 4x_2^2}$

      (ii) $\|\mathbf{x}\| = |x_1 - x_2|$

   (b) (5 pts) Using the inner product $\langle \mathbf{x}, \mathbf{y} \rangle_B = \mathbf{x}^T B \mathbf{y}$ with

$$B = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

   compute $\|\mathbf{e}_1\|_B$, $\|\mathbf{e}_2\|_B$, and $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle_B$.

   (c) (5 pts) Verify the Cauchy–Schwarz inequality for $\mathbf{e}_1$ and $\mathbf{e}_2$ with respect to $\langle \cdot, \cdot \rangle_B$.