1. The following problems are unrelated.

(a) (6pts) Write out the form of the partial fraction decomposition (do NOT solve for coefficients) for

$$g(x) = \frac{x+1}{x^2(x-3)^3(x^2+4)^2}$$

Solution:

$$\frac{x+1}{x^2(x-3)^3(x^2+4)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3} + \frac{D}{(x-3)^2} + \frac{E}{(x-3)^3} + \frac{Fx+G}{x^2+4} + \frac{Hx+I}{(x^2+4)^2}$$

(b) (12pts) Evaluate $\displaystyle\int_0^{\pi/4} x\sec^2 x\,dx$.

Solution: IBP with $u = x$, $dv = \sec^2 x\,dx$, so $du = dx$, $v = \tan x$:

$$\int_0^{\pi/4} x\sec^2 x\,dx = \Big[x\tan x\Big]_0^{\pi/4} - \int_0^{\pi/4} \tan x\,dx.$$

For the remaining integral, write $\tan x = \dfrac{\sin x}{\cos x}$ and substitute $u = \cos x$, $du = -\sin x\,dx$:

$$\int \tan x\,dx = \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos x| + C = \ln|\sec x| + C.$$

Therefore

$$\int_0^{\pi/4} x\sec^2 x\,dx = \Big[x\tan x\Big]_0^{\pi/4} - \Big[\ln|\sec x|\Big]_0^{\pi/4}.$$

At $x = 0$ both terms vanish ($\tan 0 = 0$, $\ln|\sec 0| = \ln 1 = 0$). At $x = \pi/4$: $\tan(\pi/4) = 1$, $\sec(\pi/4) = \sqrt{2}$. Hence

$$\int_0^{\pi/4} x\sec^2 x\,dx = \frac{\pi}{4} - \ln\sqrt{2} = \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

(c) (20pts) Evaluate $\displaystyle\int \frac{x^3}{\sqrt{x^2+4}}dx$

Solution:

(a) Trig substitution $x = 2\tan\theta$, $dx = 2\sec^2\theta$, $\sqrt{x^2+4} = 2\sec\theta$:

$$\int \frac{x^3}{\sqrt{x^2+4}} = \int \frac{8\tan^3\theta\,(2\sec^2\theta)}{2\sec\theta} = 8\int \tan^3\theta\sec\theta.$$

Now this is a powers-of-sec / tan integral: write $\tan^2\theta = \sec^2\theta - 1$,

$$8\int (\sec^2\theta - 1)\tan\theta\sec\theta.$$

Let $u = \sec\theta$, $du = \sec\theta\tan\theta$:

$$= 8\int (u^2-1) = 8\Big(\frac{u^3}{3} - u\Big) + C = \frac{8}{3}\sec^3\theta - 8\sec\theta + C.$$

From the reference triangle, $\sec\theta = \dfrac{\sqrt{x^2+4}}{2}$, so

$$\int \frac{x^3}{\sqrt{x^2+4}} = \frac{(x^2+4)^{3/2}}{3} - 4\sqrt{x^2+4} + C = \frac{(x^2-8)\sqrt{x^2+4}}{3} + C.$$

2. Determine whether the following integrals converge or diverge. If an integral converges you do not need to work out the value. If an integral diverges, justify why.

   (a) (12pts) $\displaystyle\int_1^\infty \frac{2+\cos x}{x^2}dx$

   (b) (12pts) $\displaystyle\int_0^2 \frac{7x^2+12}{x^2\left(x^2+4\right)}dx$

Solution:

(a) This is Type I (infinite upper limit). Since $-1 \le \cos x \le 1$, we have $1 \le 2+\cos x \le 3$, so

$$0 < \frac{2+\cos x}{x^2} \le \frac{3}{x^2} \quad \text{for } x \ge 1.$$

The comparison integral $\displaystyle\int_1^\infty \frac{3}{x^2} = 3\lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = 3$ converges ($p = 2 > 1$). By the Comparison Theorem, the given integral **converges**. (Its exact value is not elementary; convergence is all that is asked.)

(b) This is Type II improper: the integrand is undefined at $x = 0$ (lower limit). First decompose. With a repeated linear factor x^2 and the irreducible quadratic x^2+4,

$$\frac{7x^2+12}{x^2(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4}.$$

Clearing denominators: $7x^2+12 = Ax(x^2+4) + B(x^2+4) + (Cx+D)x^2$. Matching coefficients gives $A = 0,\ C = 0$ (both zero), $B = 3$, $D = 4$. Thus

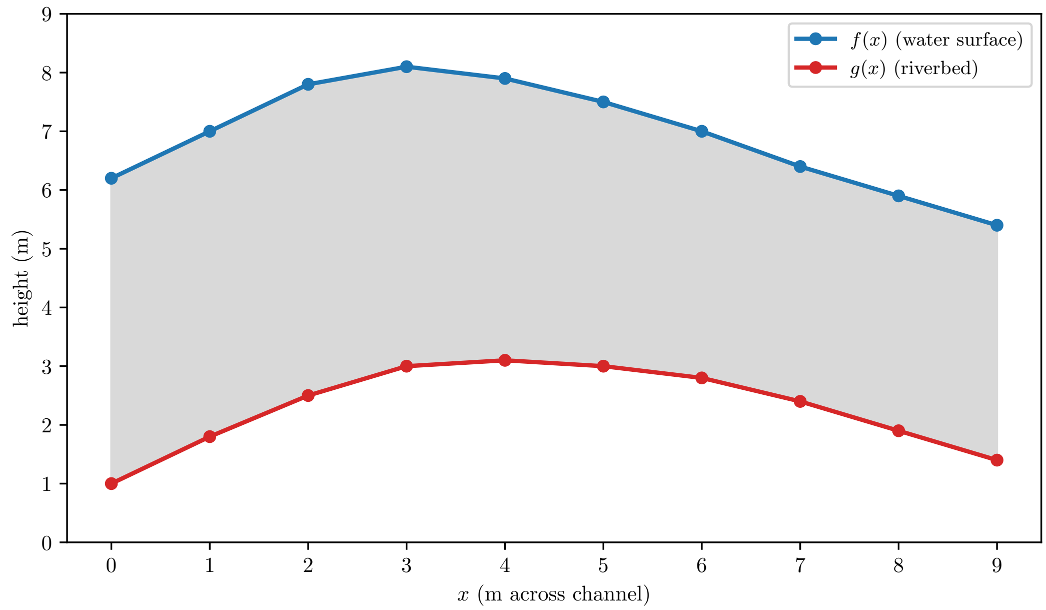
$$\frac{7x^2+12}{x^2(x^2+4)} = \frac{3}{x^2} + \frac{4}{x^2+4}.$$

Now evaluate as a limit:

$$\int_0^2 \left(\frac{3}{x^2} + \frac{4}{x^2+4}\right) = \lim_{t\to 0^+}\int_t^2 \left(\frac{3}{x^2} + \frac{4}{x^2+4}\right) = \lim_{t\to 0^+}\left[-\frac{3}{x} + 2\arctan\frac{x}{2}\right]_t^2.$$

The $-\dfrac{3}{x}$ term gives $\displaystyle\lim_{t\to 0^+}\left(-\frac{3}{2} + \frac{3}{t}\right) = +\infty$. Since this piece diverges, the integral **diverges**.

3. A surveyor measures a cross-section of a river channel at the positions $x = 0, 1, 2, \ldots, 9$ (in meters across the channel). The table records the water-surface height $f(x)$and the riverbed height $g(x)$ (in meters above a fixed reference level):



x	0	1	2	3	4	5	6	7	8	9
$f(x)$	6.2	7.0	7.8	8.1	7.9	7.5	7.0	6.4	5.9	5.4
$g(x)$	1.0	1.8	2.5	3.0	3.1	3.0	2.8	2.4	1.9	1.4

(a) (10 pts) Using the trapezoidal rule with $n = 3$, approximate the cross-sectional area of the water in the channel (You do NOT need to simplify your expression, but you should give an expression which could be easily entered into a calculator).

(b) (10 pts) A smooth function $h(x)$ is fit to the data $f(x) - g(x)$ on $[0, 9]$, and it is known that $|h''(x)| \leq \frac{1}{9}$ on the interval. How large must n be to guarantee that the Trapezoidal Rule approximation has error less than 10^{-3}? *You should solve for n, but you do not need to simplify your answer.*

Solution:

(a) For $n = 3$, $\Delta x = (9 - 0)/3 = 3$. The difference between the functions $h = f - g$ at $x = 0, 3, 6, 9$ are 5.2,5.1,4.2,4.0, so the trapezoidal approximation is,

$$T_3 = \frac{3}{2} \cdot (5.2 + 2(5.1) + 2(4.2) + 4).$$

(b) The trapezoidal error bound is $|E_T| \leq \dfrac{K(b-a)^3}{12n^2}$ with $K \geq |h''(x)|$. Take $K = \frac{1}{9}$, $a = 0$, $b = 9$:

$$|E_T| \leq \frac{(1/9)(9)^3}{12\,n^2} = \frac{81}{12\,n^2} = \frac{27}{4\,n^2}.$$

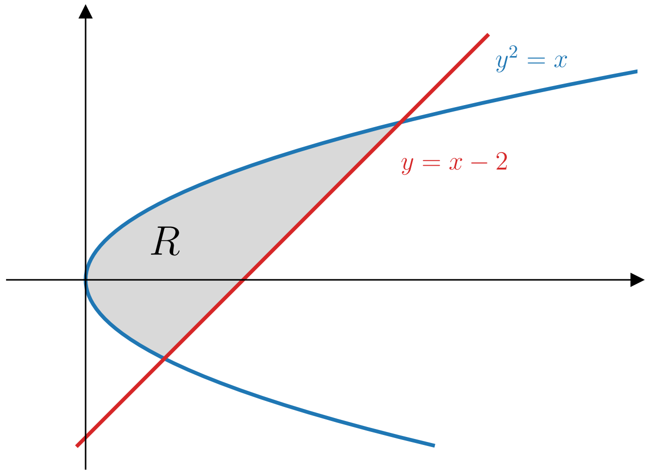
Require $\dfrac{27}{4\,n^2} < 10^{-3}$:

$$n^2 > \frac{27 \cdot 1000}{4} = \frac{27000}{4} \quad \Longrightarrow \quad n > \frac{\sqrt{27000}}{2}.$$

4. Let R be the region bounded by $y^2 = x$ and $y = x - 2$.

(a) (6pts) Sketch and shade R; label intersection points.

(b) (12pts) Set up, but do not evaluate, an integral for the volume of the solid generated by rotating R about the line $x = 4$.



Solution: First, find where the curves intersect by setting $y^2 = y + 2$:

$$y^2 - y - 2 = 0 \implies (y-2)(y+1) = 0 \implies y = -1,\ 2 \implies x = 1, 4$$

So the region runs from $y = -1$ to $y = 2$.

Since we rotate about the vertical line $x = 4$, which lies to the *right* of the region, each radius is the horizontal distance from $x = 4$ to a curve. The parabola $x = y^2$ is farther from the axis, so it gives the outer radius, while the line $x = y + 2$ gives the inner radius:

$$R = 4 - y^2, \qquad r = 4 - (y+2) = 2 - y.$$

By the washer method,

$$V = \pi \int_{-1}^{2} \left[\left(4 - y^2\right)^2 - (2 - y)^2 \right] dy.$$