1. The following questions are unrelated

   (a) (3 pts) Simplify the expression

   $\pi \sin^2\left(\frac{11\pi}{7}\right) + \pi \cos^2\left(\frac{11\pi}{7}\right)$

   Solution:

$$\begin{aligned}\pi \sin^2\left(\frac{11\pi}{7}\right) + \pi \cos^2\left(\frac{11\pi}{7}\right) &= \pi\left(\frac{11\pi}{7}\right) + \pi \cos^2\left(\frac{11\pi}{7}\right) \\ &= \pi(1) \\ &= \boxed{\pi}\end{aligned}$$

   (b) (8 pts) Consider the function g such that $g(x) = |x-5|$

   i. Is g odd, even, or neither? Please explain.

   Solution:

   $g(-x) = |-x-5| = |-(x+5)| = |x+5|$.

   So, $g(-x) \neq g(x)$ and $g(-x) \neq -g(x)$. Hence g is $\boxed{\text{Neither}}$

   ii. State the domain and range of g in interval notation. No explanation required.

   Solution:

   Domain $\boxed{(-\infty,\infty)}$ and Range $\boxed{[0,\infty)}$

2. Evaluate the following limits

   (a) (6 pts) $\lim_{x\to 0} \frac{\sec x}{\csc x}$

   Solution:

$$\begin{aligned}\lim_{x\to 0} \frac{\sec x}{\csc x} &= \lim_{x\to 0} \frac{\left(\frac{1}{\cos x}\right)}{\left(\frac{1}{\sin x}\right)} \\ &= \lim_{x\to 0} \frac{\sin x}{\cos x} \\ &= \lim_{x\to 0} \tan x \\ &= \boxed{0}\end{aligned}$$

   (b) (6 pts) $\lim_{x\to 0}\left(\frac{1}{2x} - \frac{1}{2x-x^2}\right)$

   Solution:

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{2x} - \frac{1}{2x-x^2}\right) &= \lim_{x\to 0} \frac{2x-x^2-2x}{2x(2x-x^2)} \\ &= \lim_{x\to 0} \frac{-x^2}{2x^2(2-x)} \\ &= \lim_{x\to 0} \frac{-1}{2(2-x)} \\ &= \boxed{-\frac{1}{4}}\end{aligned}$$

(c) (6 pts) $\lim_{x \to -\infty} \dfrac{x+2}{\sqrt{x^2+1}}$

Solution:

$$\begin{aligned}
\lim_{x \to -\infty} \frac{x+2}{\sqrt{x^2+1}} &= \lim_{x \to -\infty} \frac{x+2}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}} \\
&= \lim_{x \to -\infty} \frac{x+2}{\sqrt{x^2}\sqrt{\left(1+\frac{1}{x^2}\right)}} \\
&= \lim_{x \to -\infty} \frac{x+2}{|x|\sqrt{\left(1+\frac{1}{x^2}\right)}} \\
&= \lim_{x \to -\infty} \frac{x+2}{-x\sqrt{\left(1+\frac{1}{x^2}\right)}} \quad (\text{as } x \to -\infty,\ |x| = -x) \\
&= \lim_{x \to -\infty} -\frac{1+\frac{2}{x}}{\sqrt{\left(1+\frac{1}{x^2}\right)}} \\
&= -\frac{1+0}{\sqrt{1+0}} \\
&= \boxed{-1}
\end{aligned}$$

3. Let $f(x) = 2 + \dfrac{3-3x}{x^2-5x+4}$

(a) (4 pts) Write the domain of f in interval notation.
Solution:
$f(x) = 2 + \dfrac{3(1-x)}{(x-1)(x-4)}$. Hence the domain is $\boxed{(-\infty, 1) \cup (1, 4) \cup (4, \infty)}$

(b) (6 pts) Does f have any removable discontinuity? Justify your answer using limits.
Solution:

$$\begin{aligned}\lim_{x\to 1} f(x) &= 2 + \lim_{x\to 1} \frac{-3(x-1)}{(x-1)(x-4)}\\ &= 2 + \lim_{x\to 1} \frac{-3}{(x-4)}\\ &= 2+1\\ &= 3\end{aligned}$$

Hence there is a $\boxed{\text{removable discontinuity at } (1,3)}$

(c) (6 pts) Find the vertical asymptotes of f, if any. Justify using limits.
Solution:

$$\begin{aligned}\lim_{x\to 4^+} f(x) &= 2 + \lim_{x\to 4^+} \frac{-3(x-1)}{(x-1)(x-4)}\\ &= 2 + \lim_{x\to 4^+} \frac{-3}{(x-4)}\\ &= -\infty\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to 4^-} f(x) &= 2 + \lim_{x\to 4^-} \frac{-3(x-1)}{(x-1)(x-4)}\\ &= 2 + \lim_{x\to 4^-} \frac{-3}{(x-4)}\\ &= \infty\end{aligned}$$

Hence there is a $\boxed{\text{vertical asymptote at } x=4}$

(d) (6 pts) Find the horizontal asymptote(s) of f, if any. Justify using limits.
Solution:

$$\begin{aligned}\lim_{x\to \infty} f(x) &= 2 + \lim_{x\to \infty} \frac{3-3x}{x^2-5x+4}\\ &= 2 + \lim_{x\to \infty} \frac{\frac{3}{x^2}-\frac{3}{x}}{1-\frac{5}{x}+\frac{4}{x^2}}\\ &= 2 + \frac{0}{1}\\ &= 2\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to -\infty} f(x) &= 2 + \lim_{x\to -\infty} \frac{3-3x}{x^2-5x+4}\\ &= 2 + \lim_{x\to -\infty} \frac{\frac{3}{x^2}-\frac{3}{x}}{1-\frac{5}{x}+\frac{4}{x^2}}\\ &= 2 + \frac{0}{1}\\ &= 2\end{aligned}$$

Hence there is a $\boxed{\text{horizontal asymptote at } y=2}$

4. The following questions are unrelated.

   (a) (8 pts) Explain whether the following function is continuous at $x = 0$, using the definition of continuity. Please show all your work clearly

$$f(x) = \begin{cases} \dfrac{x}{\sin(3x)} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Solution:
For continuity at $x = 0$, we require that $\lim_{x \to 0} f(x) = f(0)$
Here, we have

$$\begin{aligned} \lim_{x \to 0} f(x) &= \lim_{x \to 0} \frac{x}{\sin(3x)} \\ &= \lim_{x \to 0} \frac{3x}{3\sin(3x)} \\ &= \frac{1}{3} \frac{1}{\left(\lim_{x \to 0} \frac{3x}{\sin(3x)}\right)} \quad \text{using Limit Laws} \\ &= \frac{1}{3} \cdot \frac{1}{1} = \frac{1}{3} \end{aligned}$$

whereas $f(0) = 1$. Hence, $\lim_{x \to 0} f(x) \neq f(0)$ and the function is **Not Continuous**

   (b) (8 pts) Show that the equation $\cos x = x$ has at least one solution.
Solution:
Say, $f(x) = \cos x - x$. f is continuous in $(-\infty, \infty)$, because $\cos x$ and x are.
We also have

$$f(0) = 1$$
$$f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$$

Hence, $f(x)$ changes sign in the interval $\left[0, \frac{\pi}{2}\right]$.
Hence $f(x) = 0$ has a solution in that interval, **by Intermediate Value Theorem**. In other words, $\cos x = x$ has at least one solution.

5. The following questions are unrelated.

   (a) (9 pts) Differentiate $f(x) = \sqrt{4 - x}$ using the limit definition of derivative.
Solution:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{1}{h}[\sqrt{4 - (x+h)} - \sqrt{4 - x}] \\ &= \lim_{h \to 0} \frac{4 - (x+h) - [4 - (x)]}{h(\sqrt{4 - (x+h)} + \sqrt{4 - x})} \\ &= \lim_{h \to 0} \frac{-h}{h(\sqrt{4 - (x+h)} + \sqrt{4 - x})} \\ &= \lim_{h \to 0} -\frac{1}{\sqrt{4 - (x+h)} + \sqrt{4 - x}} \\ &= \lim_{h \to 0} \frac{-1}{2\sqrt{4 - x}} \end{aligned}$$

$$f'(x) = \boxed{\frac{-1}{2\sqrt{4 - x}}}$$

(b) (8 pts) For $t \geq 0$ the position of a particle moving along the x axis is given by $x(t) = \sin t - \cos t$
What is the acceleration of the particle at the time when velocity is first equal to 0?
Solution:
Let us find the time when velocity is first equal to 0

$$\begin{aligned} v(t) &= 0 \\ \frac{dx(t)}{dt} &= 0 \\ \cos t + \sin t &= 0 \\ \sin t &= -\cos t \\ \tan t &= -1 \\ t &= \frac{3\pi}{4} \end{aligned}$$

At that point, we calculate

$$\begin{aligned} a(t) &= \frac{dv(t)}{dt} \\ a(t) &= -\sin t + \cos t \\ a\left(\frac{3\pi}{4}\right) &= -\sin\left(\frac{3\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right) \\ &= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \\ &= \boxed{-\sqrt{2}} \end{aligned}$$

6. The following questions are unrelated.

(a) (7 pts) Differentiate $f(x) = x\tan x + \frac{\sin x}{x}$
Solution:
Using Product and Quotient Rules, we obtain

$$f'(x) = \boxed{x\sec^2 x + \tan x + \frac{x\cos x - \sin x}{x^2}}$$

(b) (9 pts) Find the equation of the tangent line to the curve $y = \sin(\cos x)$ at $x = \frac{\pi}{2}$
Solution:
First we find the y coordinate corresponding to $x = \frac{\pi}{2}$

$$\begin{aligned} y|_{x=\frac{\pi}{2}} &= \sin\left(\cos(\frac{\pi}{2}\right) \\ &= \sin(0) = 0 \end{aligned}$$

Then, we find the slope m of the tangent line at $x = \frac{\pi}{2}$. Using Chain Rule,

$$\begin{aligned} y' &= \cos(\cos x)(-\sin x) \\ y'|_{x=\frac{\pi}{2}} &= -\sin\left(\frac{\pi}{2}\right)\cos\left(\cos(\frac{\pi}{2})\right) \\ &= -1 \cdot \cos(0) \\ &= -1 \end{aligned}$$

Hence, the equation of tangent line at $x = \frac{\pi}{2}$ is given by $y - 0 = -1\left(x - \frac{\pi}{2}\right)$, or

$$\boxed{y = -x + \frac{\pi}{2}}$$