**APPM 3570/STAT 3100 – Section 001** **Final Exam** **Spring 2026**

May 1, 2026

Instructor: Yu-Jui Huang

NAME: ______________________________

**Instructions:**

1. A calculator and two $3 \times 5$ note cards (one double-sided, one single-sided) are permitted.
2. Your text and other books, cell phones, and other electronic devices are not permitted during the exam.
3. Justify your answers, show all work.
4. When you have completed the exam, go to the uploading area in the room and scan your exam and upload it to Gradescope.
5. Don't forget to scan any back pages you used for extra space!
6. Verify that everything has been uploaded correctly and the pages have been associated to the correct problems.
7. Turn in your hardcopy exam.

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| On my honor as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work.<br><br>Signature: ______________________ Date: ____________ |

**Duration: 120 minutes**

**Problem 1.** (20 points) Consider $X \sim \text{Geometric}(1/2)$ and construct a random variable $Y$ as follows: given any realization $x$ of $X$, take $Y \sim \text{Poisson}(x/2)$.

(a) What is the conditional probability mass function (pmf) of $X$ given that $Y = 0$?

(b) What is the conditional expectation of $X$ given that $Y = 0$?

(c) What is the expected value of $Y$?

**Solution:**

(a) (10 points) Note that, in principle, $X$ can take any integer value greater or equal to 1. In particular, for any integer $x \geq 1$,

$$\begin{aligned} p_{X|Y}(x|y=0) &= \frac{\mathrm{P}(X=x, Y=0)}{p_Y(0)} \\ &= \frac{\mathrm{P}(X=x)\cdot \mathrm{P}(Y=0|X=x)}{p_Y(0)} \\ &= \frac{\left(\frac{1}{2}\right)^x \cdot \frac{(x/2)^0 e^{-x/2}}{0!}}{p_Y(0)} = c\cdot\left(\frac{1}{2\sqrt{e}}\right)^x, \end{aligned}$$

for some constant $c\,(=\frac{1}{p_Y(0)})$. But the constant $c$ must be such that $\sum_{x=1}^{\infty} p(x|y=0) = 1$. That is

$$c = \frac{1}{\sum_{x=1}^{\infty}\left(\frac{1}{2\sqrt{e}}\right)^x} = \frac{1-\frac{1}{2\sqrt{e}}}{\frac{1}{2\sqrt{e}}}.$$

Hence

$$p(x|y=0) = \left(1-\frac{1}{2\sqrt{e}}\right)\left(\frac{1}{2\sqrt{e}}\right)^{x-1}, \quad \text{for } x = 1, 2, \ldots$$

which we recognize as the p.m.f. of a Geometric distribution with success probability $1-\frac{1}{2\sqrt{e}}$. That is, given that $Y=0$, $X \sim \text{Geometric}\left(1-\frac{1}{2\sqrt{e}}\right)$.

(b) (4 points) In particular, since the expected value of a Geometric r.v. is the multiplicative inverse of its success probability, $\mathrm{E}[X|Y=0] = \frac{1}{1-\frac{1}{2\sqrt{e}}} = \frac{2\sqrt{e}}{2\sqrt{e}-1}$.

(c) (6 points) As $Y \mid \{X=x\} \sim \text{Poisson}(x/2)$, $\mathrm{E}[Y \mid X=x] = x/2$. That is, $\mathrm{E}[Y \mid X] = X/2$. Hence,

$$\mathrm{E}[Y] = \mathrm{E}[\mathrm{E}[Y \mid X]] = \mathrm{E}[X/2] = \frac{\mathrm{E}[X]}{2} = \frac{1/(1/2)}{2} = 1.$$

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**Problem 2.** (20 points) Let $X$ and $Y$ be independent random variables, each of which is Exponential($\lambda$). Consider the random variables $V = \frac{X}{Y}$ and $W = X + Y$.

(a) What is the joint pdf of $X$ and $Y$?

(b) Determine $X$ and $Y$ as functions of $V$ and $W$.

(c) What is the joint pdf of $V$ and $W$?

(d) Find the conditional probability density function (pdf) of $W$ given that $V = v$.

**Solution:**

(a) (3 points)

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \begin{cases} \lambda^2 e^{-\lambda(x+y)} & x > 0, \quad y > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) (3 points) $V = \frac{X}{Y}$ implies $X = VY$. Plugging this into $W = X + Y$ yields $W = (V+1)Y$. It follows that $Y = \frac{W}{V+1}$ and thus $X = VY = \frac{VW}{V+1}$.

(c) (7 points) As $(v, w) = h(x, y) = (\frac{x}{y}, x + y)$ for any $x, y > 0$,

$$J_h(x,y) = \begin{bmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{1}{y} & \frac{-x}{y^2} \\ 1 & 1 \end{bmatrix}.$$

Hence, $\det J_h(x,y) = \frac{1}{y} + \frac{x}{y^2} > 0$ and thus $|\det J_h(x,y)| = \frac{1}{y} + \frac{x}{y^2}$. It follows that

$$f_{V,W}(v,w) = \frac{f(x,y)}{|\det J_h(x,y)|} = \frac{\lambda^2 e^{-\lambda(x+y)}}{\frac{1}{y} + \frac{x}{y^2}} = \frac{\lambda^2 e^{-\lambda w}}{\frac{v+1}{w} + \frac{v(v+1)}{w}} = \frac{\lambda^2 w e^{-\lambda w}}{(v+1)^2}, \quad \text{for } v, w > 0$$

where the third equality follows from part (b).

(d) (7 points) From the joint pdf of $V$ and $W$ in part (c), we can compute

$$\begin{aligned} f_V(v) &= \int_0^\infty \frac{\lambda^2 w}{(v+1)^2} e^{-\lambda w}\, dw \quad \text{where } \lambda > 0 \\ &= \frac{\lambda^2}{(v+1)^2} \int_0^\infty w e^{-\lambda w}\, dw = \frac{\lambda^2}{(v+1)^2} \left( -\frac{w}{\lambda} e^{-\lambda w} \Big|_0^\infty + \int_0^\infty \frac{1}{\lambda} e^{-\lambda w}\, dw \right) \\ &= \frac{\lambda^2}{(v+1)^2} \left( \frac{1}{\lambda^2} \right) = \frac{1}{(v+1)^2} \end{aligned}$$

$$f_V(v) = \begin{cases} \frac{1}{(v+1)^2} & v > 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$f_{W|V}(w \mid v) = \frac{f_{V,W}(v,w)}{f_V(v)} = \lambda^2 w e^{-\lambda w}, \quad w > 0.$$

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**Problem 3.** (20 points) Let $X$ and $Y$ be i.i.d. random variables, each of which has a variance of 2. Consider

$$W = X + Y \quad \text{and} \quad Z = -329X + 514.$$

(a) Find $\mathrm{Cov}(W, Z)$.

(b) Find $\mathrm{Var}(W + Z)$.

(c) Find $\mathrm{Corr}(W, Z)$.

**Solution:**

(a) (8 points)

$$\begin{aligned}
\mathrm{Cov}(W, Z) &= \mathrm{Cov}(X + Y, -329X + 514) \\
&= \mathrm{Cov}(X + Y, -329X) + \mathrm{Cov}(X + Y, 514) \\
&= \mathrm{Cov}(X, -329X) + \mathrm{Cov}(Y, -329X) + 0 \\
&= -329\mathrm{Cov}(X, X) - 329\mathrm{Cov}(Y, X) \\
&= -329\mathrm{Var}(X) + 0 = -658
\end{aligned}$$

where the fifth inequality follows from $X$ and $Y$ being independent.

(b) (8 points) Note that $\mathrm{Var}(W + Z) = \mathrm{Var}(W) + \mathrm{Var}(Z) + 2\mathrm{Cov}(W, Z)$. As $X$ and $Y$ are independent,

$$\mathrm{Var}(W) = \mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) = 4. \tag{1}$$

On the other hand,

$$\mathrm{Var}(Z) = \mathrm{Var}(-329X + 514) = 329^2\mathrm{Var}(X) = 329^2 \cdot 2. \tag{2}$$

Hence, $\mathrm{Var}(W + Z) = 4 + 329^2 \cdot 2 + 2(-658) = 215170$.

(c) (4 points) By part (a), (1), and (2),

$$\mathrm{Corr}(W, Z) = \frac{\mathrm{Cov}(W, Z)}{\sqrt{\mathrm{Var}(W)\mathrm{Var}(Z)}} = \frac{-658}{\sqrt{4 \cdot 329^2 \cdot 2}} = -0.7071.$$

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**Problem 4.** (20 points.) Let $Y_1, Y_2, Y_3, \cdots$ be a sequence of i.i.d. random variables with pdf

$$f_Y(y) = \begin{cases} 2(1-y), & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the value of $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n Y_i^2$.

(b) Let $N$ be a discrete random variable that may take any integer value larger or equal to 1. Moreover, $N$ is independent of $Y_1, Y_2, Y_3, \cdots$ with $\mathrm{E}[N] = 12$. Determine the value of

$$\mathrm{E}\left[\sum_{i=1}^N Y_i^2\right].$$

(Hint: use conditional expectation).

**Solution:**

(a) (8 points) As $Y_1, Y_2, Y_3, \cdots$ are i.i.d., $Y_1^2, Y_2^2, Y_3^2, \cdots$ are also i.i.d. Moreover,

$$\begin{aligned} \mathrm{E}[Y_i^2] &= \int_0^1 y^2 f_Y(y)\,dy = \int_0^1 y^2\, 2(1-y)\,dy = \int_0^1 2(y^2 - y^3)\,dy \\ &= 2\left(\frac{y^3}{3} - \frac{y^4}{4}\right)\Big|_0^1 = 2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{6} \end{aligned}$$

Hence, by the law of large numbers,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n Y_i^2 = \mathrm{E}[Y_i^2] = \frac{1}{6}.$$

(b) (12 points) By the property of conditional expectation,

$$\begin{aligned} \mathrm{E}\left[\sum_{i=1}^N Y_i^2\right] = \mathrm{E}\left[\mathrm{E}\left[\sum_{i=1}^N Y_i^2 \,\middle|\, N\right]\right] &= \sum_{n=1}^\infty \mathrm{E}\left[\sum_{i=1}^N Y_i^2 \,\middle|\, N = n\right]\mathrm{P}(N=n) \\ &= \sum_{n=1}^\infty \mathrm{E}\left[\sum_{i=1}^n Y_i^2\right]\mathrm{P}(N=n) \\ &= \sum_{n=1}^\infty n\mathrm{E}[Y_i^2]\mathrm{P}(N=n) \\ &= \mathrm{E}[Y_i^2]\sum_{n=1}^\infty n\mathrm{P}(N=n) \\ &= \mathrm{E}[Y_i^2]\mathrm{E}[N] = \frac{1}{6}\cdot 12 = 2. \end{aligned}$$

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**Problem 5.** (20 points.) Use the Central Limit Theorem (CLT) to answer the following questions. Please specify clearly how and why the CLT can be applied.

(a) Consider $S \sim \text{NegativeBinomial}(20, 1/5)$. Recall that we may express $S$ as

$$S = \sum_{i=1}^{20} G_i,$$

where $G_1, \cdots, G_{20}$ are i.i.d. Geometric$(1/5)$ random variables. Approximate the probability that $S > 50$.

(b) From past experience, a professor knows that the test score of a student taking her final examination is a random variable with mean 75 and standard deviation 10. How many students will have to take the examination to ensure that with probability at least 0.95 the class average will be within 5 of 75?

**Solution:**

(a) (10 points.) As $\text{E}[G_i] = 1/(1/5) = 5$ and $\text{Var}(G_i) = [1-(1/5)]/(1/5)^2 = 20$ are finite, by the CLT, $\frac{S-20\cdot 5}{\sqrt{20\cdot 20}}$ is approximately a Normal$(0,1)$ random variable. Hence,

$$\begin{aligned}\text{P}(S > 50) &= \text{P}\left(\frac{S-100}{20} > \frac{50-100}{20}\right) \approx \text{P}(Z > -2.5), \quad \text{with } Z \sim \text{Normal}(0,1) \\ &= 1 - \Phi(-2.5) = 1 - (1-\Phi(2.5)) = \Phi(2.5) \approx 0.9938.\end{aligned}$$

(b) (10 points.) Let $X_1, \ldots, X_n$ be the grades of $n$ students. We need to determine an $n$ such that $\text{P}(70 \le \bar{X} \le 80) \ge 0.95$. Assuming the scores $X_1, \ldots, X_n$ are i.i.d., we may use the CLT to obtain the approximation:

$$\begin{aligned}\text{P}(70 \le \bar{X} \le 80) &= \text{P}\left(\frac{\sqrt{n}(70-75)}{10} \le \frac{\sqrt{n}(\bar{X}-75)}{10} \le \frac{\sqrt{n}(80-75)}{10}\right) \\ &= \text{P}\left(-\sqrt{n}/2 \le \frac{\sqrt{n}(\bar{X}-75)}{10} \le \sqrt{n}/2\right) \\ &\overset{(CLT)}{\approx} \Phi(\sqrt{n}/2) - \Phi(-\sqrt{n}/2) \\ &= 2\cdot\Phi(\sqrt{n}/2) - 1 \ge 0.95,\end{aligned}$$

i.e., need $\Phi(\sqrt{n}/2) \ge 0.975$. Since $\Phi(1.96) = 0.975$, we need $\sqrt{n}/2 \ge 1.96$, i.e., $n \ge 15.3664$. Thus, based on the CLT approximation, the professor needs at least 16 students to take the test.

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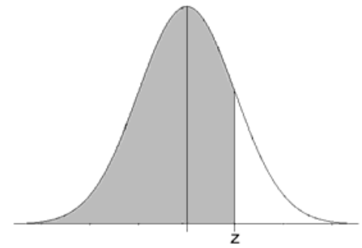
## Standard Normal Cumulative Probability Table




**Cumulative probabilities for POSITIVE z-values are shown in the following table:**

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| **0.1** | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| **0.2** | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| **0.3** | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| **0.4** | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| **0.5** | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| **0.6** | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| **0.7** | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| **0.8** | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| **0.9** | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| **1.0** | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| **1.1** | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| **1.2** | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| **1.3** | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| **1.4** | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| **1.5** | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| **1.6** | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| **1.7** | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| **1.8** | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| **1.9** | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| **2.0** | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| **2.1** | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| **2.2** | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| **2.3** | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| **2.4** | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| **2.5** | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| **2.6** | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| **2.7** | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| **2.8** | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| **2.9** | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| **3.0** | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |
| **3.1** | 0.9990 | 0.9991 | 0.9991 | 0.9991 | 0.9992 | 0.9992 | 0.9992 | 0.9992 | 0.9993 | 0.9993 |
| **3.2** | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 |
| **3.3** | 0.9995 | 0.9995 | 0.9995 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9997 |
| **3.4** | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9998 |