

April 1, 2026

Instructor: Yu-Jui Huang

NAME: _____

SECTION: 001 or 002**Instructions:**

1. A calculator and a 3×5 note card are permitted.
2. Your text and other books, cell phones, and other electronic devices are not permitted during the exam.
3. Justify your answers, show all work.
4. When you have completed the exam, go to the uploading area in the room and scan your exam and upload it to Gradescope.
5. Don't forget to scan any back pages you used for extra space!
6. Verify that everything has been uploaded correctly and the pages have been associated to the correct problems.
7. Turn in your hardcopy exam.

On my honor as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work.

Signature: _____ Date: _____

Duration: 90 minutes

Problem 1. (25 points) On average, 10 customers enter a store in a mall per hour. As the store manager, you arrive at the store at 11:00 a.m. Let N be the number of customers entering the store between 11:00 a.m. and 11:30 a.m. Let T be the amount of time (in hours) you wait until you see the first customer entering the store.

- Specify the distribution of N (i.e., the name of the distribution and its parameter(s)).
- Find the probability that there are at least 4 customers entering the store between 11:00 a.m. and 11:30 a.m.
- Specify the distribution of T (i.e., the name of the distribution and its parameter(s)).
- Find the probability that the first customer enters the shop after 11:10 a.m.
- Given that you have not yet seen the first customer at 11:10 a.m., what is the probability that the first customer will not show up until at least 11:20 a.m.?

Solution:

- (5 points.) Between 11:00 a.m. and 11:30 a.m. (half an hour), there should be on average $10/2=5$ customers entering the store. So, $N \sim \text{Poisson}(5)$.
- (5 points.)

$$\begin{aligned} P(N \geq 4) &= 1 - P(N = 0) - P(N = 1) - P(N = 2) - P(N = 3) \\ &= 1 - \frac{5^0 e^{-5}}{0!} - \frac{5^1 e^{-5}}{1!} - \frac{5^2 e^{-5}}{2!} - \frac{5^3 e^{-5}}{3!} \\ &= 1 - e^{-5} \left(1 + 5 + \frac{25}{2} + \frac{125}{6} \right) \approx 0.7350. \end{aligned}$$

- (5 points.) We know $T \sim \text{Exponential}(\lambda)$ for some $\lambda > 0$ and $\mathbb{E}[T] = \frac{1}{\lambda}$. To find the value of λ , note that since 10 customers enter the store per hour on average, the average waiting time for the next customer to show up is $1/10$ hours. That is, $\frac{1}{\lambda} = \mathbb{E}[T] = \frac{1}{10}$. Hence, $\lambda = 10$ and we conclude $T \sim \text{Exponential}(10)$.
- (5 points.) Note that “10 minutes” amounts to “ $1/6$ hours.” With $T \sim \text{Exponential}(10)$, the desired probability is

$$P(T > 1/6) = 1 - P(T \leq 1/6) = 1 - (1 - e^{-10 \cdot (1/6)}) = e^{-5/3} = 0.1889.$$

- (5 points.) By the memoryless property of the exponential distribution, the desired probability is

$$P(T > 2/6 \mid T > 1/6) = P(T > 1/6) = 0.1889,$$

where the second equality follows from part (d).

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Problem 2. (25 points.) The height of 12-month old boys is normally distributed with a mean of 78 centimeters and a variance of 9. For the following questions, you may use the table on the last page of this exam.

- (a) What is the probability that a 12-month old boy picked at random has a height between 75 and 79 centimeters?
- (b) What is approximately the 75th percentile for the height of 12-month old boys? (That is, for approximately which height are 75% of 12-month old boys shorter than this height?)
- (c) Suppose that 64 12-month old boys are picked independently at random. Let N be the total number of boys among the 64 who have a height above the 75th percentile.
 - (i) What is the distribution of N (i.e., the name of the distribution and its parameter(s))?
 - (ii) What is approximately the probability of $N = 16$? Use normal approximation to answer this question.

Solution:

- (a) (8 points.) Let X denote the height of a 12-month old boy picked at random. Then, $Z = \frac{X-78}{3} \sim Normal(0, 1)$. Let Φ denote the cdf of Z . Observe that

$$\begin{aligned}
 P(75 \leq X \leq 79) &= P\left(\frac{75-78}{3} \leq Z \leq \frac{79-78}{3}\right) \\
 &= P(-1 \leq Z \leq 1/3) \\
 &= \Phi(1/3) - \Phi(-1) \\
 &= \Phi(0.33) - (1 - \Phi(1)) \\
 &= \Phi(0.33) + \Phi(1) - 1 \\
 &= 0.6293 + 0.8413 - 1 = 0.4706,
 \end{aligned}$$

where the second last equality follows by checking the table on p. 10.

- (b) (6 points.) Let c denote the 75th percentile for the height of 12-month old boys. Then, $P(X \leq c) = 0.75$, i.e., $P(Z \leq \frac{c-78}{3}) = 0.75$. From the table on p. 10, we know $P(Z \leq 0.67) = 0.7486$ and $P(Z \leq 0.68) = 0.7517$. Hence, we can take $\frac{c-78}{3} = 0.67$ (or $\frac{c-78}{3} = 0.68$), so that $c = 80.01$ (or 80.04) centimeters.
- (c) (i) (3 points.) $N \sim Binomial(n = 64, p = 0.25)$.
- (ii) (8 points.) By taking $\mu = \mathbb{E}[N] = np = 64/4 = 16$ and $\sigma^2 = Var(N) = np(1-p) = 64 \cdot \frac{1}{4} \cdot \frac{3}{4} = 12$,

$$\frac{N-16}{\sqrt{12}} \text{ is approximated } Z \sim Normal(0, 1).$$

Hence,

$$\begin{aligned}
 P(N = 16) &= P(15.5 \leq N \leq 16.5) \\
 &\approx P\left(\frac{15.5-16}{\sqrt{12}} \leq Z \leq \frac{16.5-16}{\sqrt{12}}\right) = P\left(\frac{-1}{4\sqrt{3}} \leq Z \leq \frac{1}{4\sqrt{3}}\right) \\
 &= \Phi\left(\frac{1}{4\sqrt{3}}\right) - \Phi\left(\frac{-1}{4\sqrt{3}}\right) = 2\Phi\left(\frac{1}{4\sqrt{3}}\right) - 1 \\
 &= 2\Phi(0.14) - 1 = 2(0.5557) - 1 = 0.1114.
 \end{aligned}$$

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Problem 3. (23 points) Let X be a random variable whose cumulative distribution function (cdf) is given by

$$F(x) = \begin{cases} \frac{1}{4(1+x^2)}, & x < 0; \\ \frac{x+1}{x+4}, & x \geq 0. \end{cases}$$

- Is X discrete or continuous? Justify your answer with rigorous mathematical arguments.
- If X is discrete, determine its probability mass function (pmf). If X is continuous, determine its probability density function (pdf).
- Find the probability that $-1 \leq X \leq 2$.
- Depending on your answer in (b), determine the pmf (or pdf) of the random variable $Y = 2X - 1$.

Solution:

- (5 points) By definition, F is continuous on $(-\infty, 0)$ and $[0, \infty)$. Is it continuous at $x = 0$? Note that

$$\lim_{x \rightarrow 0^-} F(x) = \lim_{x \rightarrow 0^-} \frac{1}{4(1+x^2)} = \frac{1}{4} = F(0).$$

Hence, F is continuous at $x = 0$, and thus continuous on the whole real line \mathbb{R} . We then conclude that X is a continuous random variable.

- (5 points) The pdf of X is given by

$$f(x) = F'(x) = \begin{cases} \frac{1}{4} \frac{-2x}{(1+x^2)^2} = \frac{-x}{2(1+x^2)^2}, & \text{for } x < 0, \\ \frac{1}{4} \frac{(x+4) - (x+1)}{(x+4)^2} = \frac{3}{(x+4)^2}, & \text{for } x \geq 0. \end{cases}$$

- (5 points) Based on the cdf F given by the problem,

$$P(-1 \leq X \leq 2) = F(2) - F(-1) = \frac{2+1}{2+4} - \frac{1}{4(1+(-1)^2)} = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}.$$

- (8 points) As $y = g(x) = 2x - 1$ is a strictly increasing function on \mathbb{R} , its inverse function g^{-1} is well-defined. As $y = 2x - 1$, we obtain $x = g^{-1}(y) = \frac{y+1}{2}$. Also, we see that $x \geq 0$ if and only if $y \geq -1$. Hence, the pdf of Y is given by

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy}(g^{-1}(y)) \right| = f_X\left(\frac{y+1}{2}\right) \cdot \frac{1}{2} \\ &= \begin{cases} -\frac{1}{8} \frac{y+1}{(1+(\frac{y+1}{2})^2)^2}, & \text{for } y < -1, \\ \frac{3}{2(\frac{y+1}{2}+4)^2}, & \text{for } y \geq -1. \end{cases} \end{aligned}$$

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Problem 4. (27 points.) Let (X, Y) be a random vector in \mathbb{R}^2 uniformly distributed on the square with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.

- Find the joint probability density function (pdf) of X and Y .
- Find the marginal pdf of X and the marginal pdf of Y .
- Are X and Y independent? Justify your answer with rigorous mathematical arguments.
- Find the probability that $X \geq Y$.

Solution:

- (6 points.) As (X, Y) is uniformly distributed on the square with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$ (let us call the square the region A), it should have constant probability density $c > 0$ within the region A and 0 probability density outside of A , i.e.,

$$f_{X,Y}(x, y) = \begin{cases} c, & \text{for } (x, y) \in A, \\ 0, & \text{for } (x, y) \notin A. \end{cases}$$

Using the condition $\int \int_A f_{X,Y}(x, y) dx dy = 1$, we get $c \cdot \text{area}(A) = 1$. As $\text{area}(A) = \sqrt{2} \cdot \sqrt{2} = 2$, we have $c = 1/2$. We therefore conclude that the joint pdf of X and Y is

$$f_{X,Y}(x, y) = \begin{cases} 1/2, & \text{for } (x, y) \in A, \\ 0, & \text{for } (x, y) \notin A. \end{cases}$$

- (10 points.) The marginal pdf of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy = \begin{cases} \int_{-\infty}^{\infty} 0 dy = 0, & \text{for } x \notin [-1, 1], \\ \int_{x-1}^{1-x} \frac{1}{2} dy = 1 - x, & \text{for } x \in [0, 1], \\ \int_{-1-x}^{x+1} \frac{1}{2} dy = x + 1, & \text{for } x \in [-1, 0]. \end{cases}$$

Similarly, the marginal pdf of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx = \begin{cases} \int_{-\infty}^{\infty} 0 dx = 0, & \text{for } y \notin [-1, 1], \\ \int_{y-1}^{1-y} \frac{1}{2} dx = 1 - y, & \text{for } y \in [0, 1], \\ \int_{-1-y}^{y+1} \frac{1}{2} dx = y + 1, & \text{for } y \in [-1, 0]. \end{cases}$$

- (5 points.) For any $(x, y) \in A$ with $x, y \geq 0$, we have $f_X(x)f_Y(y) = (1-x)(1-y)$, which does not always equal $\frac{1}{2} = f_{X,Y}(x, y)$. For example, for $(x, y) = (0, 0)$, $f_X(0)f_Y(0) = 1 \cdot 1 = 1 \neq \frac{1}{2} = f_{X,Y}(0, 0)$. Hence, X and Y are not independent.
- (6 points.) By plotting the square A and the straight line $y = x$ on the \mathbb{R}^2 , we see that $y = x$ separates A into equal halves, i.e., $A_1 = A \cap \{(x, y) : x \geq y\}$ and $A_2 = A \cap \{(x, y) : x < y\}$. Hence,

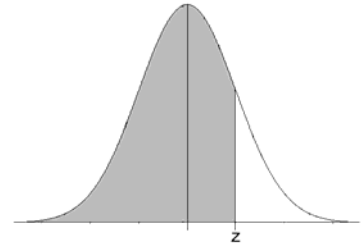
$$P(X \geq Y) = \int \int_{A_1} f_{X,Y}(x, y) dx dy = \frac{1}{2} \cdot \text{area}(A_1) = \frac{1}{2} \cdot \frac{1}{2} \text{area}(A) = \frac{1}{4} \cdot 2 = \frac{1}{2}.$$

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Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998