Numerical Analysis Preliminary Exam
1 to 4 PM, May 21, 2026

Instructions: Submit solutions to four (and no more) of the following six problems. Please start each problem on a new page. You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. Write your Student ID on your exam; **do not write your name on your exam**.

Problem 1: Rootfinding

Consider the following iteration

$$x_{n+1} = \ln \frac{1}{\epsilon} - \ln x_n,\ x_0 = \ln \frac{1}{\epsilon}.$$

A fixed point of this iteration is a value $x = \alpha$ such that

$$\epsilon\alpha = e^{-\alpha}.$$

Find an interval of values of $\epsilon > 0$ for which the iteration converges.

Problem 2: Interpolation & Approximation

(a) Let p be a trigonometric polynomial

$$p(x) = \sum_{n=-M}^{M} c_n e^{\mathrm{i}nx}$$

that interpolates a function $f(x)$ at the points

$$x_m = 2\pi \frac{m}{2M+1},\ m = 0, \ldots, 2M$$

i.e. $p(x_m) = f(x_m)$ for $m = 0, \ldots, 2M$. Derive an explicit expression for the coefficients c_n in terms of the values of f at the interpolation nodes.

(b) Suppose that f is a 2π periodic function with uniformly-convergent Fourier series

$$f(x) = \sum_{-\infty}^{\infty} \hat{f}_k e^{\mathrm{i}kx}.$$

Derive an explicit expression for the error $\hat{f}_n - c_n$ in terms of the Fourier coefficients $\hat{f}_n$ of f.

Problem 3: Quadrature

The following is a generalization of numerical quadrature (as it uses evaluations of $f(x)$ and $f'(x)$) to approximate integrals of smooth functions on the interval $(-1, 1)$

$$I[f] = \int_{-1}^{1} f(x)dx \approx Q[f] = \sum_{j=0}^{n} \omega_j f(x_j) + \eta_j f'(x_j)$$

(a) Consider the case $n = 1$, corresponding to evaluation at two points $x_0, x_1 \in [-1, 1]$. Assuming we have already chosen our quadrature nodes x_0 and x_1, write down a system of equations aimed at *integrating a polynomial of the highest degree possible q exactly.*

(b) Given a choice of x_0, x_1, consider the Hermite interpolant basis of cubic polynomials $\{H_0, H_1, K_0, K_1\}$ satisfying

$$\begin{array}{llll} H_0(x_0) = 1 & H_0(x_1) = 0 & H_0'(x_0) = 0 & H_0'(x_1) = 0 \\ H_1(x_0) = 0 & H_1(x_1) = 1 & H_1'(x_0) = 0 & H_1'(x_1) = 0 \\ K_0(x_0) = 0 & K_0(x_1) = 0 & K_0'(x_0) = 1 & K_0'(x_1) = 0 \\ K_1(x_0) = 0 & K_1(x_1) = 0 & K_1'(x_0) = 0 & K_1'(x_1) = 1 \end{array}$$

Applying the quadrature rule to each member of this basis (or by some other method), find formulas for $\omega_0, \omega_1, \eta_0, \eta_1$.

(c) Using the fact that you can integrate polynomials up to degree q exactly (or by some other method), write down an explicit expression for the quadrature error $E[f] = I[f] - Q[f]$; simplify as much as you can. Your expression should not include the integral of f.

Problem 4: Linear Algebra

Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ symmetric matrices. Given $\mathbf{C} \in \mathbb{R}^{n \times n}$, we pose the *Sylvester Equation*:

$$\mathbf{AX} - \mathbf{XB} = \mathbf{C}$$

for unknown matrix $\mathbf{X} \in \mathbb{R}^{n \times n}$. This equation shows up in problems in control theory, Lyapunov stability in dynamical systems, among others.

(a) Explain why, given our assumptions, we can compute eigendecompositions:

$$\mathbf{A} = \mathbf{URU}^T \quad \mathbf{B} = \mathbf{VSV}^T$$

with **U,V** orthogonal matrices (their columns are orthonormal bases of eigenvectors) and **R,S** real, diagonal matrices (diagonal entries are eigenvalues). Write a simple description of the most efficient algorithm you know to compute these decompositions.

(b) Plugging in both eigendecompositions (or by some other method), show the Sylvester equation is equivalent to

$$\mathbf{RY} - \mathbf{YS} = \mathbf{D}$$

with $\mathbf{Y} = \mathbf{U}^T\mathbf{XV}$ and $\mathbf{D} = \mathbf{U}^T\mathbf{CV}$.

(c) By evaluating the k-th column of the identity in (b), write down a formula for the k-th column of $\mathbf{Y}$, $\boldsymbol{y}_k$. Show that this equation has a unique solution if and only if $\mathbf{A}$ and $\mathbf{B}$ *do not have any eigenvalues in common.*

Problem 5: Ordinary Differential Equations

(a) Write the ordinary differential equation

$$y''' + 2y'' + y' + 2y = 0$$

with initial conditions

$$y(0) = 1,\ y'(0) = 0,\ y''(0) = 0$$

as a first-order system, and find its characteristic polynomial.

(b) Sketch the absolute stability region for the second-order explicit Runge-Kutta method (find its intersections with real and imaginary axes and include these in your plot),

$$\begin{aligned} k_1 &= hf(t_n, y_n) \\ k_2 &= hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\ y_{n+1} &= y_n + k_2 \end{aligned}$$

(c) For the fourth-order explicit Runge Kutta method

$$\begin{aligned} k_1 &= hf(t_n, y_n) \\ k_2 &= hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\ k_3 &= hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \\ k_4 &= hf\left(t_n + h, y_n + k_3\right) \\ y_{n+1} &= y_n + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4 \end{aligned}$$

write equations to determine the absolute stability region and use the equations to check that points $z = 2\mathrm{i}$ and $z = -2$ are inside it.

(d) For the ODE from (a), should you use the method from parts (b) or (c)? Explain your answer.

Problem 6: Partial Differential Equations

(a) Show that the explicit Dufort-Frankel method

$$\frac{u_j^{m+1} - u_j^{m-1}}{2h_t} = \frac{c}{h_x^2}\left(u_{j+1}^m - u_j^{m+1} - u_j^{m-1} + u_{j-1}^m\right),$$

for the heat equation

$$u_t = cu_{xx},\ \ c > 0,\ \ t \geq 0,$$

with periodic boundary conditions is second order in both space and time variables, is conditionally consistent (i.e. convergence depends on the ratio of time/space step sizes). Here u_j^m is the value at the space step j and time step m , h_x is the step size in space and h_t is the step size in time variables.

(b) Analyze the stability of the Dufort-Frankel method: Provide an algebraic equation whose solution determines stability, and state the condition on the solution of this algebraic equation that ensures that the method is stable.