1. (25 points) Problems (a) and (b) are unrelated.

   (a) Consider the quadric surface given by

   $$9x^2 - y^2 - 2y + 3z^2 - 12z + 2 = 0.$$

   i. Determine the standard equation of the surface.

   ii. Identify the type of surface.

   (b) Consider plane A described by $x + 2y = 1$, and plane B described by $2z - x = 2$. Find the equation of the plane that is orthogonal to planes A and B, and contains the point $(1, 1, 1)$. Your final answer should be in the form $ax + by + cz = d$.

Solution:

(a) We will begin by completing the square for each variable, collect constants, and then divide by 9 to get a standard form:

$$\begin{aligned}
9x^2 - y^2 - 2y + 3z^2 - 12z + 2 &= 0 \\
9x^2 - (y^2 + 2y + 1) + 3(z^2 - 4z + 4) &= -1 + 3(4) - 2 \\
9x^2 - (y+1)^2 + 3(z-2)^2 &= 9 \\
x^2 - \frac{(y+1)^2}{9} + \frac{(z-2)^2}{3} &= 1.
\end{aligned}$$

This is a Hyperboloid of One Sheet.

(b) The normal vectors to planes A and B are:

$$\mathbf{n}_1 = \langle 1, 2, 0 \rangle, \quad \mathbf{n}_2 = \langle -1, 0, 2 \rangle.$$

A plane orthogonal to both has normal vector:

$$\mathbf{n}_3 = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} = \langle 4, -2, 2 \rangle.$$

Using the point $(1, 1, 1)$, the plane equation is:

$$\mathbf{n}_3 \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

$$\langle 4, -2, 2 \rangle \cdot \langle x - 1, y - 1, z - 1 \rangle = 0.$$

Expanding, we have

$$4(x-1) - 2(y-1) + 2(z-1) = 0$$

which becomes

$$2x - y + z = 2.$$

2. (25 points) Suppose D is the region in the xy-plane that is between the circles of radii 1 and 3 centered at the origin in the lower half plane, $y \leq 0$. Let $f(x,y) = \dfrac{1}{x^2+y^2}$.

   (a) Sketch the region D on a clearly labeled xy-plane. Be sure to label all axes and intersection points on your sketch, and to shade in the region.

   (b) We will consider the integral $\iint_D f(x,y)\,dA$ by first setting it up two different ways, and then integrating just one of these two.

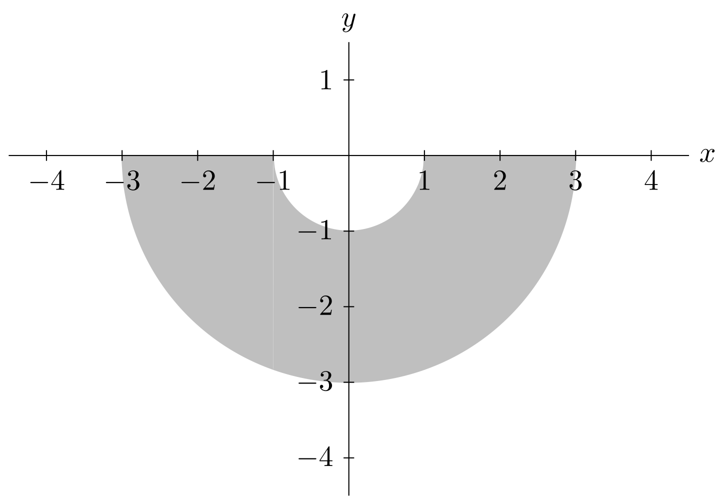
      i. Set up **but do not evaluate** the integral (or sum of integrals) $\iint_D f(x,y)\,dA$ using rectangular coordinates.

      ii. Set up **but do not evaluate** the integral (or sum of integrals) $\iint_D f(x,y)\,dA$ using polar coordinates.

      iii. Evaluate the expression you obtained in (i) **or** the expression you obtained in (ii). (If you evaluate both, clearly indicate which one you want to be graded.)

Solution:

(a) We have the following sketch:



(b) i. Some reasonable choices include the following:

$$\int_{-3}^{-1}\int_{-\sqrt{9-x^2}}^{0} \frac{1}{x^2+y^2}\,dy\,dx + \int_{-1}^{1}\int_{-\sqrt{9-x^2}}^{-\sqrt{1-x^2}} \frac{1}{x^2+y^2}\,dy\,dx + \int_{1}^{3}\int_{-\sqrt{9-x^2}}^{0} \frac{1}{x^2+y^2}\,dy\,dx$$

OR

$$2\left(\int_{0}^{1}\int_{-\sqrt{9-x^2}}^{-\sqrt{1-x^2}} \frac{1}{x^2+y^2}\,dy\,dx + \int_{1}^{-3}\int_{-\sqrt{9-x^2}}^{0} \frac{1}{x^2+y^2}\,dy\,dx\right)$$

OR

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{0} \frac{1}{x^2+y^2}\,dy\,dx - \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0} \frac{1}{x^2+y^2}\,dy\,dx$$

ii.

$$\int_{\pi}^{2\pi}\int_{1}^{3}\frac{1}{r^2}\,r\,dr\,d\theta$$

iii. The expression in polar coordinates is significantly easier to evaluate:

$$\begin{aligned}\int_{\pi}^{2\pi}\int_{1}^{3}\frac{1}{r^2}\,r\,dr\,d\theta &= \left(\int_{\pi}^{2\pi} d\theta\right)\left(\int_{1}^{3}\frac{1}{r}\,dr\right)\\ &= (2\pi-\pi)(\ln 3-\ln 1)\\ &= \pi\ln 3.\end{aligned}$$

3. (25 points) Consider the force given by

$$\mathbf{F}(x,y,z)=\langle 2x, e^z, ye^z\rangle.$$

(a) Determine if the force, $\mathbf{F}$, is irrotational throughout $\mathbb{R}^3$.

(b) Determine if the force, $\mathbf{F}$, is incompressible throughout $\mathbb{R}^3$.

(c) Find the work done by $\mathbf{F}$ in moving an object from $(1,1,1)$ to $(2,-1,\ln 3)$. *If you use a Vector Calculus (Chapter 13) theorem, state its name and justify why it is applicable.*

Solution:

(a) We see that the curl of $\mathbf{F}$ is

$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x & e^z & ye^z\end{vmatrix}=\langle e^z-e^z, 0-0, 0-0\rangle=\mathbf{0}.$$

So, $\mathbf{F}$ is irrotational throughout $\mathbb{R}^3$.

(b) We see that the divergence of $\mathbf{F}$ is

$$\nabla\cdot\mathbf{F}=\frac{\partial}{\partial x}(2x)+\frac{\partial}{\partial y}(e^z)+\frac{\partial}{\partial z}(ye^z)=2+ye^z.$$

So, $\mathbf{F}$ is **not** incompressible throughout $\mathbb{R}^3$.

(c) The path that the object takes is not mentioned here, but that is fine since the vector field is conservative (because the vector field is irrotational and $\mathbb{R}^3$ is simply connected), meaning the line integral is path independent. That is, we can find a potential function for $\mathbf{F}$ and apply the Fundamental Theorem of Line Integrals. We see that a potential function is $f(x,y,z)=x^2+ye^z$. (Note that we have $\nabla f=\mathbf{F}$.)

So, we have that the work is

$$\begin{aligned}\int_{C_2}\mathbf{F}\cdot d\mathbf{r} &= \int_{C_2}\nabla f\cdot d\mathbf{r}\\ &= f(2,-1,\ln 3)-f(1,1,1)\\ &= (2^2-e^{\ln 3})-(1+(1)e^1)\\ &= -e.\end{aligned}$$

4. (25 points) Consider the vector field

$$\mathbf{F}(x,y,z) = \langle x^3, y^3, z^3 \rangle$$

and the surface S that is the boundary of the region in space given by

$$x^2 + y^2 \leq 1, \qquad 0 \leq z \leq 2.$$

Assume S has *outward orientation*.

(a) Use Gauss' Divergence theorem to calculate the flux of $\mathbf{F}$ through S.

(b) Let S_1 denote the surface that is the side (cylinder) *and* the bottom disk ($z = 0$) of S, but *not* the top disk given by $z = 2$. Use your answer from (a) along with **only one additional surface integral** to calculate the flux of $\mathbf{F}$ through S_1.

Solution:

(a) Let E be the solid region described in the problem statement, By the Divergence theorem

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \nabla \cdot \mathbf{F}\, dV &= \iiint_E (3x^2 + 3y^2 + 3z^2)\, dV \\
&= 3 \int_0^{2\pi} \int_0^1 \int_0^2 (r^2 + z^2) r\, dz\, dr\, d\theta \\
&= 3 \int_0^{2\pi} \int_0^1 2r^3 + \frac{8}{3} r\, dr\, d\theta \\
&= \frac{11}{2} \int_0^{2\pi} d\theta \\
&= 11\pi.
\end{aligned}$$

(b) Let S_2 be the top disk. So, $S = S_1 \cup S_2$. We need

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS - \iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS = 11\pi - \iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS.$$

We need only compute the latter surface integral for S_2. We can parameterize with

$$\mathbf{r}(x,y) = \langle x, y, 2 \rangle$$

for (x,y) in the unit circle, which we will label R. We have

$$\mathbf{r}_x \times \mathbf{r}_y = \langle 1,0,0 \rangle \times \langle 0,1,0 \rangle = \langle 0,0,1 \rangle,$$

which is pointing in the correct direction for the top disk, up. We also have

$$\mathbf{F}(\mathbf{r}(x,y)) = \langle x^3, y^3, 8 \rangle.$$

So,

$$\begin{aligned}
\iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_R \mathbf{F}(\mathbf{r}(x,y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y)\, dA \\
&= 8 \iint_R dA
\end{aligned}$$

$$= 8\pi,$$

where the last integral gives the area of the unit circle.

(Alternatively for the top disk, we can note that the normal vector is just $\mathbf{k}$. Since the z-component of $\mathbf{F}$ is constant, 8, we get the flux through S_2 equals $8 \times \text{Area}(S_2) = 8\pi$.)

Thus, we have

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS = 11\pi - 8\pi = 3\pi.$$

5. (25 points) Let

$$\mathbf{F}(x, y, z) = \langle xy,\ xz,\ x^2 \rangle.$$

Let S be the surface given by

$$z = \frac{1}{2}\left(x^2 + y^2\right), \quad 0 \le z \le 8.$$

Assume the *surface is oriented upward*.

(a) Set up and simplify, **but do NOT evaluate**, an integral that equals the flux of $\nabla \times \mathbf{F}$ through S. Your answer should be an appropriate integral in terms of an appropriate surface parameterization with a simplified integrand and clear limits of integration that would be ready to integrate (but again, you don't have to actually evaluate the integral).

(b) Use one of our Vector Calculus theorems (Chapter 13) to compute the flux of $\nabla \times \mathbf{F}$ through S. *State the name of the theorem that you use.* (Hint: You should write numerical quantities in a factored form to simplify your evaluation.)

Solution:

(a) We see that the curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xz & x^2 \end{vmatrix} = \langle -x, -2x, z - x \rangle.$$

We can parameterize the surface with

$$\mathbf{r}(r, \theta) = \left\langle r\cos\theta, r\sin\theta, \frac{1}{2}r^2 \right\rangle, \quad 0 \le r \le 4, \quad \le \theta \le 2\pi.$$

The upper bound on r comes from converting $8 = \frac{1}{2}(x^2 + y^2)$ to polar and solving for r. Then, we have

$$\mathbf{r}_r \times \mathbf{r}_\theta = \langle \cos\theta, \sin\theta, r \rangle \times \langle -r\sin\theta, r\cos\theta, 0 \rangle = \langle -r^2\cos\theta, -r^2\sin\theta, r \rangle.$$

Note that the z-coordinate is positive, so this is the correct normal vector.

Also, we have that in terms of the parameterization the curl of $\mathbf{F}$ is

$$\langle -r\cos\theta, -2r\cos\theta, \frac{1}{2}r^2 - r\cos\theta \rangle.$$

Thus, we have

$$\begin{aligned}
\iint_S \nabla \times \mathbf{F}\, dS &= \int_0^{2\pi}\int_0^4 (\nabla \times \mathbf{F})(\mathbf{r}(r,\theta) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^4 \langle r\cos\theta, -2r\cos\theta, \frac{1}{2}r^2 - r\cos\theta\rangle \cdot \langle -r^2\cos\theta, -r^2\sin\theta, r\rangle\, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^4 -r^3\cos^2\theta + 2r^3\sin\theta\cos\theta + \frac{1}{2}r^3 - r^2\cos\theta\, dr\, d\theta.
\end{aligned}$$

(b) We will use Stokes' Theorem and compute the flow along the boundary of S, which is the circle given by $8 = \frac{1}{2}(x^2 + y^2)$ and $z = 8$. We can use the parameterization

$$\mathbf{r}(t) = \langle 4\cos t, 4\sin t, 8\rangle, \qquad 0 \le t \le 2\pi$$

which is a counterclockwise orientation when viewed from above, consistent with the orientation of the surface.

We have

$$\mathbf{r}'(t) = \langle -4\sin t, 4\cos t, 0\rangle$$

and

$$\mathbf{F}(\mathbf{r}(t)) = \langle 2^4\sin t\cos t, 2^5\cos t, 2^4\cos^2 t\rangle.$$

So, we have

$$\begin{aligned}
\iint_S \nabla \times \mathbf{F}\, dS &= \int_C \mathbf{F} \cdot \mathbf{T}\, ds \\
&= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_0^{2\pi} -2^6\sin^t\cos t + 2^7\cos^2 t\, dt \\
&= 2^6\int_0^{2\pi} -\sin^t\cos t + 1 + \cos(2t)\, dt \\
&= 2^6[-\sin^3 t + t + \frac{1}{2}\sin(2t)]_0^{2\pi} \\
&= 2^7\pi.
\end{aligned}$$

6. (25 points) An astronaut is on the surface of a planet given by $x^2 + y^2 + 4z^2 = 9$. The temperature in this region of space is given by $T(x, y, z) = 2xy + 4z$.

(a) What quadric surface is the surface of the planet?

(b) Find the hottest point(s) and coldest point(s) on the surface of the planet, and the temperatures at these points. (Clearly indicate at which points the temperature is the hottest and at which points the temperature is the coldest.)

(c) Our astronaut is equipped with a drill and wants to find the hottest point(s) and coldest point(s) on the entire planet, including its interior. Would you advise the astronaut to stay on the surface of the planet, or to drill to a point inside the planet? Explain your reasoning.

Solution:

(a) The surface of the planet is an ellipsoid, which we can write in standard form as

$$\frac{x^2}{9} + \frac{y^2}{9} + \frac{4z^2}{9} = 1.$$

(b) Write the surface of the planet as a constraint of the form

$$g(x, y, z) = x^2 + y^2 + 4z^2 = 9.$$

Using Lagrange Multipliers $\nabla g = \lambda \nabla T$, we get the system of equations

$$\langle 2x, 2y, 8z\rangle = \nabla g = \lambda \nabla T = \langle 2y, 2x, 4\rangle.$$

That is,

$$\begin{aligned} x &= \lambda y, \\ y &= \lambda x, \\ 2z &= \lambda. \end{aligned}$$

Combining the first and second equations above, we have $x = \lambda^2 x$; this equation is satisfied if either $x = 0$ or $\lambda = \pm 1$, so we will split into these cases.

Case $x = 0$. In this case, we have both $x = 0$ and $y = 0$, which means $z = \pm 3/2$. Plugging these into the temperature, we get

$$\begin{aligned} T(0, 0, 3/2) &= 6, \\ T(0, 0, -3/2) &= -6. \end{aligned}$$

Case $\lambda = 1$. If $\lambda = 1$, then $z = 1/2$ and $x = y$. Plugging these into $g(x, y, z) = 9$ gives $x = \pm 2$. This gives two solutions $(2, 2, 1/2)$ and $(-2, -2, 1/2)$. The temperatures here are

$$\begin{aligned} T(2, 2, 1/2) &= 10, \\ T(-2, -2, 1/2) &= 10. \end{aligned}$$

Case $\lambda = -1$. If $\lambda = -1$, then $z = -1/2$ and $y = -x$. Plugging into $g(x, y, z) = 9$ gives again $x = \pm 2$. We have two solutions $(2, -2, -1/2)$ and $(-2, 2, -1/2)$. The temperatures here are

$$\begin{aligned} T(2, -2, -1/2) &= -10, \\ T(-2, 2, -1/2) &= -10. \end{aligned}$$

The hottest points on the surface of the planet are at $(2, 2, 1/2)$ and $(-2, -2, 1/2)$ with temperature $T = 10$. The coldest points on the surface of the planet are at $(2, -2, -1/2)$ and $(-2, 2, -1/2)$ with temperature $T = -10$.

(c) Since T is continuous and the entire planet $x^2 + y^2 + 4z^2 \leq 9$ is closed and bounded, we know from the extreme value theorem that the extreme values of the temperature occur either at critical points on the boundary (which we already obtained in the previous part) or at critical points of T inside. So, we just need to check for any critical points inside the planet. Since T is differentiable, this would occur where $\nabla T = \mathbf{0}$. Computing the gradient gives

$$\nabla T(x, y, z) = \langle 2y, 2x, 4\rangle,$$

which is non-zero for any choice of (x, y, z). We should advise the astronaut to stay on the surface of the planet.