1. (24 pts) Evaluate the following integrals:

   (a) $\displaystyle\int_{-\infty}^{0} xe^x\,dx$

   (b) $\displaystyle\int \sin^5\theta\cos^3\theta\,d\theta$

   Solution:

   (a) Using integration by parts, with $u = x$ and $dv = e^x dx$, we have

$$\begin{aligned}\int_{-\infty}^{0} xe^x\,dx &= \lim_{t\to-\infty} xe^x\big|_t^0 - \int_{-\infty}^{0} e^x\,dx\\ &= \lim_{t\to-\infty} te^t - \big(\lim_{s\to-\infty}(1-e^s)\big)\\ &= \lim_{t\to-\infty}\frac{t}{e^{-t}} - 1\\ &\overset{LH}{=} \lim_{t\to-\infty}\frac{1}{-e^{-t}} - 1\\ &= 0 - 1\\ &= -1\end{aligned}$$

   (b)

$$\begin{aligned}\int \sin^5\theta\cos^3\theta\,d\theta &= \int \sin^5\theta\cos^2\theta\cos\theta\,d\theta\\ &= \int \sin^5\theta(1-\sin^2\theta)\cos\theta\,d\theta\\ &= \int(\sin^5\theta - \sin^7\theta)\cos\theta\,d\theta\end{aligned}$$

   setting $u = \sin\theta$ and $du = \cos\theta\,d\theta$, we get

$$\begin{aligned}&= \int(u^5 - u^7)\,du\\ &= \frac{u^6}{6} - \frac{u^8}{8} + C\\ &= \frac{\sin^6\theta}{6} - \frac{\sin^8\theta}{8} + C.\end{aligned}$$

2. (24 pts) Determine whether the following quantities converge or diverge. Remember to state any tests or theorems you use, and check their conditions.

   (a) $\{a_n\}_{n=3}^{\infty}$, where $a_n = \dfrac{\sin(n^2+1)\cos(n)}{n-2}$

   (b) $\displaystyle\sum_{n=0}^{\infty}\frac{(n+3)!}{\sqrt{n+1}\cdot 3^n}$

   Solution:

(a) We know $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$. Then,

$$-1 \leq \sin(n^2+1)\cos(n) \leq 1 \implies -\frac{1}{n-2} \leq \frac{\sin(n^2+1)\cos(n)}{n-2} \leq \frac{1}{n-2}.$$

Taking the limit of the compound inequality with $n \to \infty$ yields

$$0 \leq \lim_{n\to\infty} \frac{\sin(n^2+1)\cos(n)}{n-2} \leq 0.$$

By the squeeze theorem, we must have

$$\boxed{\lim_{n\to\infty} \frac{\sin(n^2+1)\cos(n)}{n-2} = 0.}$$

(b) Using the Ratio Test,

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+4)!}{\sqrt{n+2}\cdot 3^{n+1}} \frac{\sqrt{n+1}\cdot 3^n}{(n+3)!}\right| = \lim_{n\to\infty} \left|\frac{(n+4)}{3}\sqrt{\frac{n+1}{n+2}}\right| = \infty \cdot 1 = \infty.$$

Since $L = \infty > 1$, the series $\boxed{\text{diverges.}}$

3. (20 pts) Consider the region in Quadrant 1 which lies between the curves

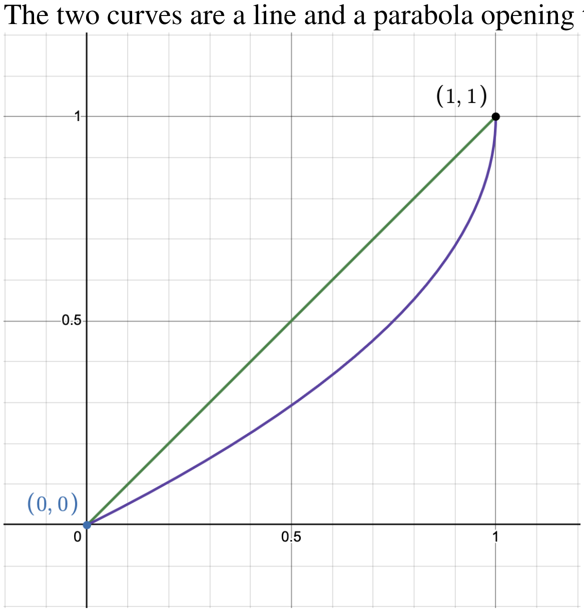
$$x = 2y - y^2,$$
$$x = y.$$

(a) Sketch this region, and label any points of intersection.

(b) Set up, but do not evaluate, an integral which gives the volume of the solid generated by revolving this region around $y = -1$.

Solution:

(a) The two curves are a line and a parabola opening to the left:



To find where the two curves intersect, we set their x values equal and solve for y:

$$y = 2y - y^2$$

$$0 = y - y^2$$
$$0 = y(1 - y)$$

They intersect when $y = 0$ and $y = 1$. The x values for the points are found by using $y = x$, so the points are

$$(0, 0) \text{ and } (1, 1)$$

(b) We use the cylindrical shell method to calculate the volume. As we are rotating about a line parallel to the x-axis, we integrate in y. We start with the basic integral:

$$V = \int_c^d 2\pi r h(y) dy$$

Using the intersection points above to get the limits of integration and the rotation axis of $y = -1$ we have

$$V = \int_0^1 2\pi(1 + y)h(y) dy$$

Since $x = y$ is to the left of the parabola $x = 2y - y^2$ we subtract the line from the parabola to get the $h(y)$ function:

$$V = \int_0^1 2\pi(1 + y)(2y - y^2 - y) dy$$
$$V = \int_0^1 2\pi(1 + y)(y - y^2) dy$$

4. (20 pts) Consider the integral $\displaystyle\int_0^1 \ln\left(\frac{1}{2}x + 1\right) dx$. All parts of this problem refer to this integral.

(a) Use the Maclaurin series for $\ln(x + 1)$ to find a series representation for this integral (make sure to evaluate the bounds; your answer should not have an integral sign in it).

(b) Use the Alternating Series Estimation theorem to determine the error bound if the first four nonzero terms from (a) are used to approximate the value of the integral.

Solution:

(a) First recall the Maclaurin series for $\ln(u + 1)$ (sometimes known as the Mercator series)

$$\ln(u + 1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{u^n}{n}, \ -1 < u \le 1.$$

Now let $u = x/2$

$$\ln\left(\frac{x}{2} + 1\right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \left(\frac{x}{2}\right)^n, \ -1 < \frac{x}{2} \le 1.$$

The series converges on the interval $-2 < x \le 2$, and the integral in question is over $0 < x < 1$, so we can integrate the series term by term. Note that

$$\int_0^1 \left(\frac{x}{2}\right)^n dx = \frac{1}{2^n} \left[\frac{x^{n+1}}{n + 1}\right]_{x=0}^{x=1} = \frac{1}{(n + 1)2^n}.$$

The series representation of the integral is therefore

$$\int_0^1 \ln\left(\frac{1}{2}x + 1\right) dx = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n(n + 1)2^n}.$$

(b) To use the ASET we need to first verify its conditions, which are

- The series alternates;
- The absolute value of the terms in the series decreases monotonically;
- The terms in the series limit to 0 as $n \to \infty$.

The first condition is clearly met because $[n(n+1)2^n]^{-1} > 0$ and the third condition is also clearly met

$$\lim_{n\to\infty} \frac{1}{n(n+1)2^n} = 0.$$

To check monotonic decrease, note first that

$$n^2 + 5n + 4 > 0$$

for $n \geq 1$ since it is a sum of positive terms when $n \geq 1$. Adding $n(n+1) = n^2 + n$ to both sides yields

$$2n^2 + 6n + 4 > n(n+1).$$

Note that the left side simplifies to $2n^2 + 6n + 4 = 2(n+1)(n+2)$. Multiplying both sides by 2^n preserves the inequality because $2^n > 0$, and produces

$$2^{n+1}(n+1)(n+2) > 2^n n(n+1).$$

For $n \geq 1$ both sides of the inequality are positive, so we can divide the inequality by both sides, which produces

$$\frac{1}{n(n+1)2^n} > \frac{1}{(n+1)(n+2)2^{n+1}}.$$

For $b_n = \left|(-1)^{n-1}\frac{1}{n(n+1)2^n}\right|$ this shows that $b_n > b_{n+1}$, which is monotone decrease.

Having established that we can use the ASET to bound the error in the truncated series we recall that the error bound is simply the magnitude of the next term in the series. In this case

$$\left|\int_0^1 \ln\left(\frac{1}{2}x + 1\right)\,dx - \left(\frac{1}{4} - \frac{1}{24} + \frac{1}{96} - \frac{1}{320}\right)\right| \leq \frac{1}{5\times(5+1)\times 2^5} = \frac{1}{960}.$$

5. (24 pts) Suppose a bird's position is given by the equations

$$x(t) = t^3 + t + 1$$
$$y(t) = \sqrt{t},$$

for $0 \leq t \leq 10$, where $x(t)$ is the bird's horizontal position and $y(t)$ is the bird's height off the ground at time t.

(a) At which (x, y) point does the bird start? When does the bird land on the ground again in the given time interval? Give a specific time, or say "never" with appropriate justification.

(b) What is the slope of the tangent line to the bird's path at $t = 2$?

(c) Set up, but do not evaluate, an integral which gives the distance the bird travels over this time interval. In other words, your integral should give the length of the path in the xy-plane.

Solution:

(a) The fird starts at $\boxed{(x(0), y(0)) = (1, 0)}$. The bird lands on the ground anytime $y = 0$ meaning $y = \sqrt{t} = 0$ which implies $t = 0$. Since $t = 0$ marks the start of the flight, the bird $\boxed{\text{never}}$ touches the ground again.

(b) We can find the slope via

$$\left.\frac{dy}{dx}\right|_{t=2} = \left.\frac{y'(t)}{x'(t)}\right|_{t=2} = \left.\frac{\frac{1}{2\sqrt{t}}}{3t^2+1}\right|_{t=2} = \frac{\frac{1}{2\sqrt{2}}}{13} = \boxed{\frac{1}{26\sqrt{2}}}.$$

(c) To compute the distance the bird has travelled, we need to compute the arclength of the path. We need the length element for arc length,

$$ds = \sqrt{(x')^2 + (y')^2}\,dt = \sqrt{(3t^2+1)^2 + \left(\frac{1}{2\sqrt{t}}\right)^2}\,dt = \sqrt{(3t^2+1)^2 + \frac{1}{4t}}\,dt.$$

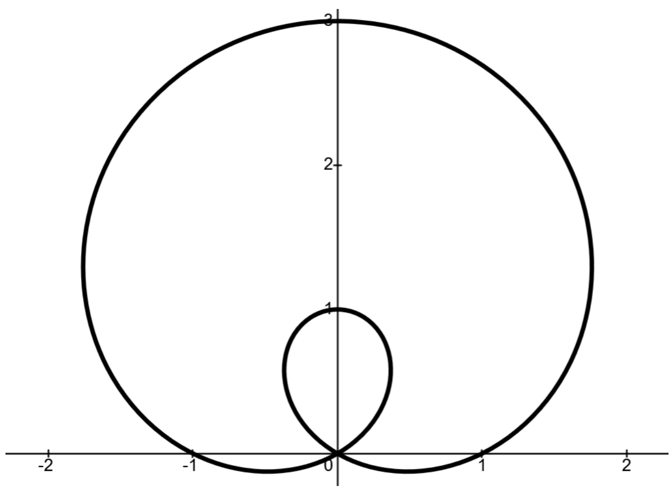
Then, the distance is given by

$$\boxed{L = \int_a^b ds = \int_0^{10} \sqrt{(3t^2+1)^2 + \frac{1}{4t}}\,dt.}$$

6. (22 pts) Consider the polar curve given by

$$r = 1 + 2\sin\theta, \qquad \text{for } 0 \le \theta \le 2\pi,$$

which is plotted here:



(a) Give an expression for the slope of a tangent line to this curve as a function of θ.

(b) Set up, but do not evaluate, an integral that gives the area of the inner loop.

Solution:

(a)

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Method 1: Use the general formula for dy/dx in terms of r and θ, which can be derived as follows:

Since r is a function of θ, apply the product rule to derive expressions for $dy/d\theta$ and $dx/d\theta$:

$$\frac{dy}{d\theta} = \frac{d}{d\theta}[r\sin\theta] = r\cdot\cos\theta + \sin\theta\cdot\frac{dr}{d\theta}$$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}[r\cos\theta] = r\cdot(-\sin\theta) + \cos\theta\cdot\frac{dr}{d\theta}$$

So, the general formula for dy/dx in terms of r and θ is:

$$\frac{dy}{dx} = \frac{r\cos\theta + \sin\theta\cdot\dfrac{dr}{d\theta}}{-r\sin\theta + \cos\theta\cdot\dfrac{dr}{d\theta}}$$

Since $r = 1 + 2\sin\theta$, we have $\dfrac{dr}{d\theta} = 2\cos\theta$.

Therefore, the slope of a tangent line to the specified curve is given by the following expression:

$$\frac{dy}{dx} = \frac{r\cos\theta + \sin\theta\cdot(2\cos\theta)}{-r\sin\theta + \cos\theta\cdot(2\cos\theta)} = \boxed{\frac{(1+2\sin\theta)\cos\theta + 2\sin\theta\cos\theta}{-(1+2\sin\theta)\sin\theta + 2\cos^2\theta}}$$

Method 2: Derive expressions for x and y in terms of only the variable θ, as follows:

$$y = r\sin\theta = (1+2\sin\theta)\sin\theta$$
$$x = r\cos\theta = (1+2\sin\theta)\cos\theta$$

The product rule can be applied to derive expressions for $dy/d\theta$ and $dx/d\theta$:

$$\frac{dy}{d\theta} = (1+2\sin\theta)\cos\theta + \sin\theta\cdot(2\cos\theta)$$

$$\frac{dx}{d\theta} = (1+2\sin\theta)\cdot(-\sin\theta) + \cos\theta\cdot(2\cos\theta)$$

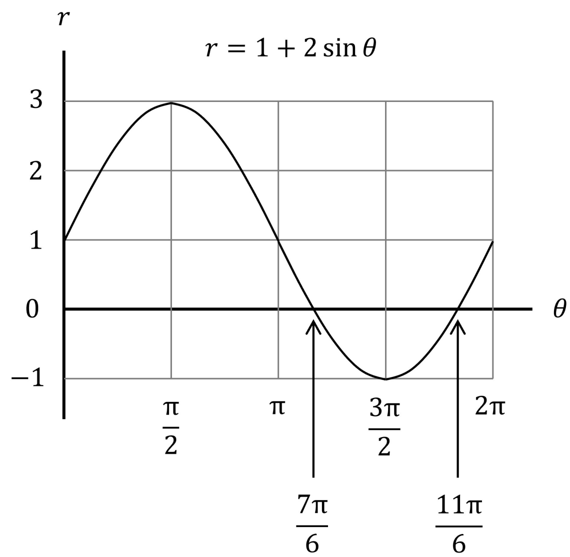
Therefore, the slope of a tangent line to the specified curve is given by the following expression:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \boxed{\frac{(1+2\sin\theta)\cos\theta + 2\sin\theta\cos\theta}{-(1+2\sin\theta)\sin\theta + 2\cos^2\theta}}$$

(b) The formula for the area of the region bounded by the polar curve $r = r(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$, where $0 < \beta - \alpha < 2\pi$, is:

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

The graph of $r = 1 + 2\sin\theta$ in Cartesian coordinates is given by:



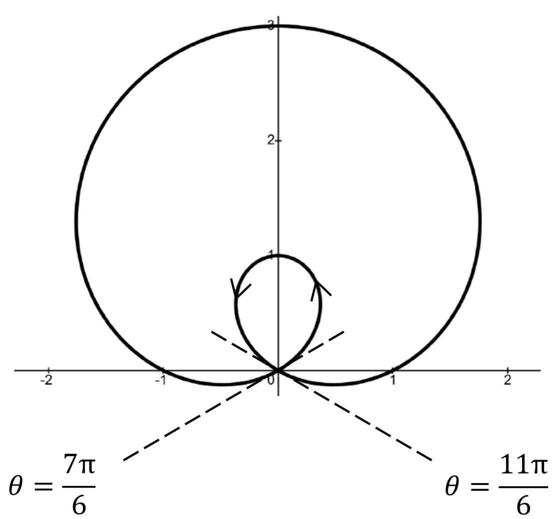
The two values of θ on the interval $[0, 2\pi]$ for which $r = 0$ are determined as follows:

$$r = 1 + 2\sin\theta = 0$$

$$\sin\theta = -1/2$$

$$\theta = 7\pi/6, 11\pi/6 \quad \text{(as indicated on the preceding graph)}$$

Since the value of r is negative between $\theta = 7\pi/6$ and $\theta = 11\pi/6$, the polar curve corresponding to that interval is above the horizontal axis instead of below it. In particular, it is that interval of θ values that corresponds to the inner loop of the given curve, as shown in the following polar graph:



Therefore, the value of the following integral represents the area of the inner loop of the polar curve:

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta = \boxed{\int_{7\pi/6}^{11\pi/6} \frac{1}{2} (1 + 2\sin\theta)^2 \, d\theta}$$

7. (16 pts) Consider the conic section given by the equation

$$x^2 + 5y^2 + 10y + 4x = 1.$$

(a) Determine whether this conic section is a parabola, ellipse, or hyperbola.

(b) If you said **ellipse** or **hyperbola** for (a), determine the center of the conic section. If you said **parabola**, determine its vertex.

Solution:

(a) We must complete the squares for both x and y:

$$\begin{aligned}
x^2 + 5y^2 + 10y + 4x &= 1 \\
(x^2 + 4x) + 5(y^2 + 2y) &= 1 \\
(x^2 + 4x + 4 - 4) + 5(y^2 + 2y + 1 - 1) &= 1 \\
(x^2 + 4x + 4) - 4 + 5(y^2 + 2y + 1) - 5 &= 1 \\
(x + 2)^2 + 5(y + 1)^2 &= 10 \\
\frac{(x + 2)^2}{10} + \frac{(y + 1)^2}{2} &= 1
\end{aligned}$$

Since both of the terms are positive, we see that this is an ellipse.

(b) We read off the center of the ellipse from the completed squares: $(-2, -1)$