

1. (28 points) The following parts are not related. You may use any method to find the derivatives. Show all supporting work and box your answer.

(a) Find $f'(x)$ when $f(x) = \frac{\sin(\tan(x))}{x}$.

(b) Use logarithmic differentiation to find the derivative of $y = (x^2 + 7)^x$. Write your answer in terms of x . *You do not need to simplify your answer.*

(c) Find $g'(x)$ when $g(x) = xe^{5x} + \sinh(x)$.

(d) Find dy/dx if $\ln(xy) = e^{x-y}$

Solution:

(a) Using the quotient rule and the chain rule, we have

$$f'(x) = \frac{x \cdot \cos(\tan(x)) \sec^2(x) - \sin(\tan(x)) \cdot 1}{x^2}$$

(b) To begin using the process of logarithmic differentiation, we shall take the natural log of each side, then take the derivative with respect to x . Finally, we will solve for y' and make any substitutions necessary to put our final answer in terms of x .

$$\begin{aligned} \ln(y) &= \ln(x^2 + 7)^x \\ &= x \ln(x^2 + 7) \end{aligned}$$

Now, we take the derivative of each side with respect to x :

$$\begin{aligned} \frac{d}{dx} (\ln(y)) &= \frac{d}{dx} (x \ln(x^2 + 7)) \\ \frac{1}{y} y' &= x \cdot \frac{1}{x^2 + 7} \cdot (2x) + \ln(x^2 + 7) \\ y' &= y \cdot \left(\frac{2x^2}{x^2 + 7} + \ln(x^2 + 7) \right) \\ y' &= (x^2 + 7)^x \cdot \left(\frac{2x^2}{x^2 + 7} + \ln(x^2 + 7) \right) \end{aligned}$$

Thus,
$$y' = (x^2 + 7)^x \cdot \left(\frac{2x^2}{x^2 + 7} + \ln(x^2 + 7) \right)$$

(c) To find $g'(x)$, we will use the rules for derivatives of exponentials and hyperbolic trig functions. We will also use the product rule.

$$g'(x) = 5xe^{5x} + e^{5x} + \cosh(x)$$

(d) Find dy/dx if $\ln(xy) = e^{x-y}$

$$\begin{aligned}\frac{1}{xy}(xy' + y) &= (1 - y')e^{x-y} \\ \frac{y'}{y} + \frac{1}{x} &= e^{x-y} - y'e^{x-y} \\ \frac{y'}{y} + y'e^{x-y} &= e^{x-y} - \frac{1}{x} \\ y' \left(\frac{1}{y} + e^{x-y} \right) &= e^{x-y} - \frac{1}{x} \\ y' \left(\frac{1 + ye^{x-y}}{y} \right) &= \frac{xe^{x-y} - 1}{x} \\ y' &= \frac{y(xe^{x-y} - 1)}{x(1 + ye^{x-y})}\end{aligned}$$

Therefore, $y' = \frac{y(xe^{x-y} - 1)}{x(1 + ye^{x-y})}$.

2. (40 points) The following parts are unrelated. Show all supporting work and box your answers.

(a) Evaluate the limit: $\lim_{x \rightarrow \infty} x^3 e^{-x}$.

(b) Evaluate the limit: $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$.

(c) Evaluate the indefinite integral: $\int \sec^3 x \tan x dx$.

(d) Evaluate the definite integral: $\int_1^2 \frac{x^2}{x^3 + 1} dx$

Solution:

(a) We will use L'Hopital's Rule here.

$$\begin{aligned} \lim_{x \rightarrow \infty} x^3 e^{-x} &= \lim_{x \rightarrow \infty} \frac{x^3}{e^x} \\ &\stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{3x^2}{e^x} \\ &\stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{6x}{e^x} \\ &\stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{6}{e^x} \\ &= 0 \end{aligned}$$

Therefore, $\boxed{\lim_{x \rightarrow \infty} x^3 e^{-x} = 0}$.

(b) Note that if we were to attempt direct substitution here, we'd have $0/0$. Note also that L'Hopital's rule is an acceptable way to compute this limit. The solution shown below uses an algebraic technique from chapter 1:

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} &= \lim_{x \rightarrow 3} \frac{(x + 3)(x - 3)}{x - 3} \\ &= \lim_{x \rightarrow 3} x + 3 \\ &= 6 \end{aligned}$$

Therefore, $\boxed{\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = 6}$.

(c) To evaluate this indefinite integral, we will use u-substitution. Let

$$\begin{aligned} u &= \sec(x) \\ du &= \sec(x) \tan(x) dx \end{aligned}$$

Using this substitution, we have:

$$\begin{aligned} \int \sec^3 x \tan x dx &= \int u^2 du \\ &= \frac{1}{3} u^3 + C \end{aligned}$$

$$= \frac{1}{3} \sec^3(x) + C$$

Therefore, $\boxed{\int \sec^3 x \tan x dx = \frac{1}{3} \sec^3(x) + C}$.

(d) We will again use u-substitution to solve this definite integral.

$$\begin{aligned} u &= x^3 + 1 \\ du &= 3x^2 dx \end{aligned}$$

We use this u-substitution to transform the upper and lower bounds. if $x = 1$, then $u = 1^3 + 1 = 2$. If $x = 2$, then $u = 2^3 + 1 = 9$.

$$\begin{aligned} \int_1^2 \frac{x^2}{x^3 + 1} dx &= \frac{1}{3} \int_2^9 \frac{1}{u} du \\ &= \frac{1}{3} \ln |u| \Big|_2^9 \\ &= \frac{1}{3} (\ln(9) - \ln(2)) \end{aligned}$$

Therefore, $\boxed{\int_1^2 \frac{x^2}{x^3 + 1} dx = \frac{1}{3} (\ln(9) - \ln(2))}$.

3. (15 points) For what values of a and b is the following function $f(x)$ continuous at $x = 0$? Fully support your answer using the definition of continuity, which includes evaluating the appropriate limits. Box your answers.

$$f(x) = \begin{cases} \frac{\sin(5x)}{x} & , \quad x < 0 \\ a & , \quad x = 0 \\ b(x-2)^2 & , \quad x > 0 \end{cases}$$

Solution:

First we will investigate the conditions for continuity from the left.

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} \frac{\sin(5x)}{x} \\ &= \lim_{x \rightarrow 0^-} \frac{5}{5} \cdot \frac{\sin(5x)}{x} \\ &= 5 \lim_{x \rightarrow 0^-} \frac{\sin(5x)}{5x} \\ &= 5 \cdot 1 \end{aligned}$$

Since $f(0) = a$, for continuity to hold from the left at 0, we need $\boxed{a = 5}$.

Next, we will find the limit from the right:

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} b(x-2)^2 \\ &= b(-2)^2 \\ &= 4b \end{aligned}$$

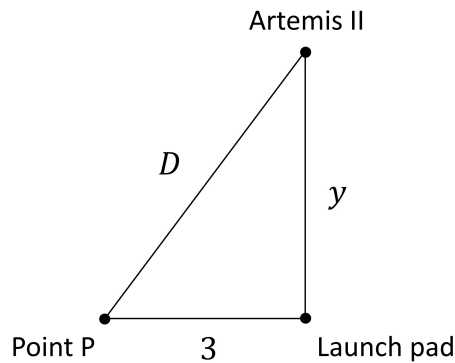
Since $f(0) = 5$, we must have that $4b = 5$, which implies that $\boxed{b = 5/4}$.

4. (15 points) Suppose the Artemis II rocket is traveling directly upward from a launch pad, and a point P is located on the ground 3 km from the launch pad. Assume the ground between the launch pad and point P is flat and horizontal.

If the rocket is traveling at 20 km per minute when it is 4 km directly above the launch pad, how fast is the straight-line distance between the rocket and point P changing at that moment? Show all work and include the correct unit of measurement in your final answer.

Solution:

Let $y = y(t)$ represent the vertical height of the Artemis rocket above the launch pad and let $D = D(t)$ represent the straight-line distance between the Artemis rocket and point P, as depicted below.



By the Pythagorean Theorem, $D^2 = 3^2 + y^2$. Implicit differentiation can be applied as follows.

$$D^2 = 3^2 + y^2$$

$$2D \frac{dD}{dt} = 2y \frac{dy}{dt}$$

$$D \frac{dD}{dt} = y \frac{dy}{dt}$$

When the rocket is 4 km directly above the launch pad, we have the following:

$$y = 4$$

$$\frac{dy}{dt} = 20$$

$$D = \sqrt{3^2 + 4^2} = 5$$

Therefore,

$$5 \cdot \frac{dD}{dt} = 4 \cdot 20$$

$$\frac{dD}{dt} = \boxed{16 \text{ km per minute}}$$

5. (28 points) Parts (a), (b), and (c) are unrelated. Clearly show your supporting work and box your answer.

(a) For $f(x) = \int_0^{3x} \sin^{-1}(t)dt$, find $f'(x)$.

(b) For $g(x) = \tan^{-1}(x^2)$, find $g'(x)$.

(c) Suppose $h(x)$ is such that $h'(x) = \frac{1}{(1-x)^2}$ and $h''(x) = \frac{2}{(1-x)^3}$

(i) On what interval(s) is $h(x)$ increasing?

(ii) On what interval(s) is $h(x)$ concave down?

Solution:

(a) Using the Fundamental Theorem of Calculus, we have that,

$$f'(x) = \frac{d}{dx} \int_0^{3x} \sin^{-1}(t)dt = \boxed{3 \sin^{-1}(3x)}$$

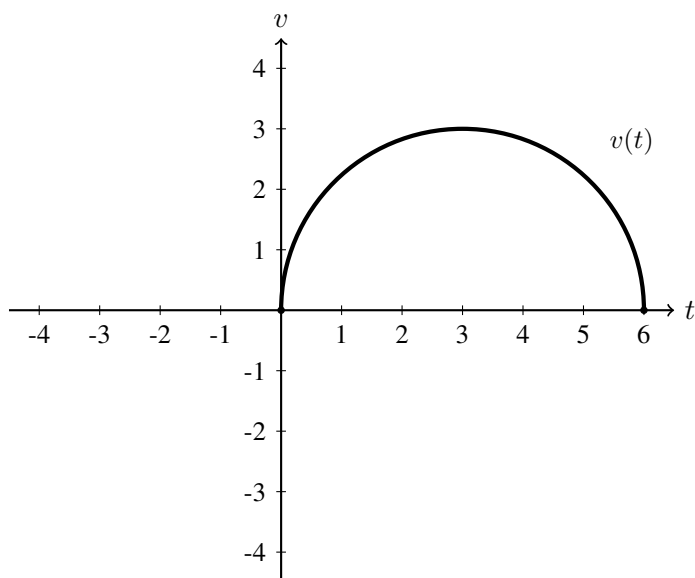
(b)

$$g'(x) = \frac{1}{1+(x^2)^2} \cdot (2x) = \boxed{\frac{2x}{1+x^4}}$$

(c) (i) $h(x)$ is increasing when $h'(x) > 0$. Since $h'(x) = \frac{1}{(1-x)^2} > 0$ for all x except for at $x = 1$ as $h'(1)$ DNE, then $h(x)$ is increasing on $\boxed{(-\infty, 1) \cup (1, \infty)}$.

(ii) $h(x)$ is concave down when $h''(x) < 0$. Since $h''(x) = \frac{2}{(1-x)^3}$, this second derivative will be negative when $(1-x)^3 < 0 \implies x > 1$. Therefore, $h(x)$ is concave down on $\boxed{(1, \infty)}$.

6. (24 points) The airspeed velocity (m/s) of an unladen European swallow is given by the function graphed below. You may assume that the swallow is traveling in a straight horizontal line. You may assume that the graph below is a half-circle. Answer the following questions. No justification is required for the following questions. Include relevant units in your answers.



- What is the distance that the swallow travels from time $t = 0$ to $t = 6$?
- What is the maximum velocity of the swallow on the given time interval: $[0, 6]$?
- Over which open interval is the swallow accelerating?
- At what point(s) in time is the swallow moving at a constant speed? (i.e. $a(t) = 0$)
- At what point(s) in time is the swallow at rest?

Solution:

- Since $v(t)$ is non-negative over the interval of time from $[0, 6]$ seconds, the distance traveled is equivalent to the area under $v(t)$.

$$\text{distance} = \int_0^6 v(t) dt = \frac{1}{2} \pi (3)^2 = \boxed{\frac{9\pi}{2} \text{ meters}}$$

- The magnitude of the maximum velocity (speed) achieved is $\boxed{3 \text{ m/s}}$ according to the graph.
- The swallow is accelerating when $v'(t) > 0$. Notice that $v(t)$ has a positive slope when $t \in (0, 3)$.
 $\boxed{(0, 3) \text{ seconds}}$
- $\boxed{t = 3 \text{ seconds}}$
- $\boxed{t = 0 \text{ and } t = 6 \text{ seconds}}$