1. [2360/042726 (25 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a two-column table (letter - answer) completely separate from any work you do to arrive at the answer.

   (a) An undamped oscillator with circular frequency $\omega_0 = 3$ is forced by $10t\sin 3t$. The guess for the particular solution when using the method of undetermined coefficients is $x_p = (At + B)\sin 3t + (Ct + D)\cos 3t$.

   (b) If $\vec{\mathbf{b}} \in \text{Col}\,\mathbf{A}$, where $\mathbf{A}$ has only positive eigenvalues, then there exists a unique vector $\vec{\mathbf{x}}$ such that $\vec{\mathbf{x}} - \mathbf{A}^{-1}\vec{\mathbf{b}} = \vec{\mathbf{0}}$.

   (c) If $\mathbf{A}$ is nonsingular, then $\left|\mathbf{A}\mathbf{A}^{\mathrm{T}}\mathbf{A}^{-1}\right| = |\mathbf{A}|$.

   (d) If $f_y(t_0, y_0)$ is not defined, then $y' = f(t, y),\ y(t_0) = y_0$ cannot have a unique solution.

   (e) Euler's method cannot be used to approximate the solution of $y' = y^{-1},\ y(1) = 0$ even though the initial value problem possesses a unique solution.

   (f) The set, $\mathbb{W}$, of points in $\mathbb{R}^2$ lying on either the x-axis or y-axis is a subspace of $\mathbb{R}^2$.

   (g) The system $\begin{aligned} x' &= x^4 + y^2 + 1 \\ y' &= x + y - 1 \end{aligned}$ possesses at least one equilibrium solution.

   (h) If $a > 0$, solutions to $y'' + \sqrt{a}y' + ay = 0$ all approach 0 as $t \to \infty$.

   (i) $\mathbb{S} = \text{span}\,\{t, \cos t\}$ is the solution space of $(t\cos t - \sin t)\,y'' + (t\sin t)\,y' - (\sin t)\,y = 0$.

   (j) There exists a curve in the xy-plane such that if solutions to the differential equation $e^y\,(y' + 1) = 2e^x$ cross this curve, they do so with a slope of 1.

SOLUTION:

   (a) **FALSE** $x_p = (At^2 + Bt)\sin 3t + (Ct^2 + Dt)\cos 3t$.

   (b) **TRUE** A vector $\vec{\mathbf{b}}$ is in the column space of a matrix $\mathbf{A}$ if and only if $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$. Since $\mathbf{A}$ has only positive eigenvalues, it is invertible. Thus, a unique vector $\vec{\mathbf{x}}$ exists such that $\vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$ which is equivalent to $\vec{\mathbf{x}} - \mathbf{A}^{-1}\vec{\mathbf{b}} = \vec{\mathbf{0}}$.

   (c) **TRUE** $\left|\mathbf{A}\mathbf{A}^{\mathrm{T}}\mathbf{A}^{-1}\right| = |\mathbf{A}||\mathbf{A}^{\mathrm{T}}||\mathbf{A}^{-1}| = |\mathbf{A}||\mathbf{A}||\mathbf{A}|^{-1} = |\mathbf{A}|$

   (d) **FALSE** Picard's theorem says nothing about this situation. There may or may not be a unique solution.

   (e) **TRUE** Euler's method is $y_1 = hy_0^{-1} = \frac{h}{0}$ which is not defined.

   (f) **FALSE** The set is not closed under addition: $(1, 0) \in \mathbb{W}$ and $(0, 1) \in \mathbb{W}$ but $(1, 0) + (0, 1) = (1, 1) \notin \mathbb{W}$.

   (g) **FALSE** $x^4 + y^2 + 1$ is never zero. Thus the system has no v nullclines and thus can have no equilibrium points.

   (h) **TRUE** The characteristic equation is $r^2 + \sqrt{a}r + a = 0$ with complex roots given by $\frac{-\sqrt{a}\pm\sqrt{a-4a}}{2} = \frac{-\sqrt{a}\pm i\sqrt{3a}}{2}$ having negative real part. Basis for the solution space is $\left\{e^{-\sqrt{a}t/2}\cos\frac{\sqrt{3a}}{2}t, e^{-\sqrt{a}t/2}\sin\frac{\sqrt{3a}}{2}t\right\}$.

   (i) **FALSE** $W[t, \cos t] = -t\sin t - \cos t$ so the functions are linearly independent and there are two of them, enough to potentially be a basis. However,

$$(t\cos t - \sin t)\,(t)'' + (t\sin t)\,(t') - (\sin t)\,(t) = t\sin t - t\sin t = 0 \implies t \text{ is a solution}$$

$$(t\cos t - \sin t)\,(\cos t)'' + (t\sin t)\,(\cos t)' - (\sin t)\,(\cos t) = (t\cos t - \sin t)\,(-\cos t) - t\sin^2 t - \sin t\cos t = -t \neq 0$$

   This means $\cos t$ is not a solution. Thus $\mathbb{S} \neq \text{span}\,\{t, \cos t\}$

   (j) **TRUE** The equation can be rewritten as $y' = 2e^{x-y} - 1$. Setting $2e^{x-y} - 1 = 1$ yields $x - y = 0$ or $y = x$, which is the isocline corresponding to slope 1 of the solution.

■

2. [2360/042726 (17 pts)] Wildlife biologists have determined that the population of hummingbirds in a certain region is growing logistically. At the same time, these biologists have noted that a parasitic infection is killing the birds at a constant rate of 49 thousand birds per year. The differential equation governing this scenario is $p'(t) = (14 - p)p - 49$. $p(t)$ gives the number of hummingbirds (in thousands) at time t (years).

   (a) (2 pts) Draw a properly labeled phase line for the equation and state the stability of all equilibrium solutions.

   (b) (3 pts) Are there initial hummingbird populations, p_0, that will tend to a nonzero steady state as t goes to infinity? If so, find those initial populations as well as the nonzero steady state which they will approach. If not, explain why not. Hint: you need not solve the differential equation to answer this.

(c) (12 pts) If there are initially 6000 hummingbirds $[p(0) = 6]$ will the population sustain itself or will the birds go extinct in a finite amount of time? If the birds go extinct, when will that occur? If they don't, explain why not.

SOLUTION:

(a) We have $p' = 14p - p^2 - 49 = -(p^2 - 14p + 49) = -(p-7)^2$. The following phase line shows that the equilibrium solution $p = 7$ is semistable.

$$\downarrow$$

7 ◓

$$\downarrow$$

(b) Yes. From part (a), if $p_0 \geq 7$, solutions will tend to 7.

(c) We need to solve the initial value problem $p' = -(p-7)^2$, $p(0) = 6$, which is separable.

$$\int \frac{\mathrm{d}p}{(p-7)^2} = \int -dt$$

$$-(p-7)^{-1} = -t + C$$

$$\frac{1}{p-7} = t - C$$

$$p(t) = \frac{1}{t-C} + 7$$

$$p(0) = 6 = \frac{1}{-C} + 7 \implies C = 1$$

$$p(t) = \frac{1}{t-1} + 7$$

To check for sustainability or extinction, can we find a t_0 with $0 < t_0 < \infty$ such that $p(t_0) = 0$?

$$0 = \frac{1}{t_0 - 1} + 7$$

$$-7 = \frac{1}{t_0 - 1}$$

$$t_0 = 1 - \frac{1}{7} = \frac{6}{7}$$

The hummingbirds will go extinct in $\frac{6}{7}$ years.

■

3. [2360/042726 (15 pts)] Find the general solution of the linear system consisting of the three equations

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= -1 \\ 6x_2 - 18x_4 &= 12. \\ 3x_1 - 2x_3 &= 7 \end{aligned}$$

Solve this system by finding the RREF of an appropriate matrix. Write your answer using the Nonhomogeneous Principle. In addition, find the solution space of the associated homogeneous system and state its dimension. Hints: create the appropriate matrix very carefully before starting; the correct answer does not contain fractions; zero credit for using a method other than the RREF.

SOLUTION:

$$\begin{bmatrix}1 & -2 & -1 & 3\\ 0 & 6 & 0 & -18\\ 3 & 0 & -2 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-1\\12\\7\end{bmatrix} \implies \left[\begin{array}{cccc|c}1 & -2 & -1 & 3 & -1\\ 0 & 6 & 0 & -18 & 12\\ 3 & 0 & -2 & 0 & 7\end{array}\right] \begin{array}{l} R_2^*=\frac{1}{6}R_2 \\ R_3^*=-3R_1+R_3\end{array} \left[\begin{array}{cccc|c}1 & -2 & -1 & 3 & -1\\ 0 & 1 & 0 & -3 & 2\\ 0 & 6 & 1 & -9 & 10\end{array}\right] \begin{array}{l} R_1^*=2R_2+R_1 \\ R_3^*=-6R_1+R_3\end{array}$$

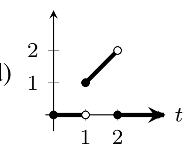
$$\left[\begin{array}{cccc|c}1 & 0 & -1 & -3 & 3\\ 0 & 1 & 0 & -3 & 2\\ 0 & 0 & 1 & 9 & -2\end{array}\right] R_1^*=1R_3+R_1 \left[\begin{array}{cccc|c}1 & 0 & 0 & 6 & 1\\ 0 & 1 & 0 & -3 & 2\\ 0 & 0 & 1 & 9 & -2\end{array}\right] \implies \begin{array}{l} x_1 = 1-6t \\ x_2 = 2+3t \\ x_3 = -2-9t \\ x_4 = t\end{array}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = t\begin{bmatrix}-6\\3\\-9\\1\end{bmatrix} + \begin{bmatrix}1\\2\\-2\\0\end{bmatrix},\ t \in \mathbb{R}$$

The solution space of the associated homogeneous system is span$\left\{\begin{bmatrix}-6\\3\\-9\\1\end{bmatrix}\right\}$ having dimension 1. ■

4. [2360/042726 (16 pts)] For parts (a) and (b) perform the operations. For parts (c) and (d), write $f(t)$ using step functions.

(a) $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}(s-1)}{(s-1)^2+b^2}\right\}$ (b) $\mathscr{L}\left\{(2-3t)\,\text{step}(t-1)\right\}$ (c) $f(t) = \begin{cases} 2 & t<3 \\ -4 & 3 \le t < 4 \\ 1 & t \ge 4\end{cases}$ (d)



SOLUTION:

(a)

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}(s-1)}{(s-1)^2+b^2}\right\} = \mathscr{L}^{-1}\left\{e^{-2s}\left(\left.\frac{s}{s^2+b^2}\right|_{s\to s-1}\right)\right\} = \text{step}(t-2)\mathscr{L}^{-1}\left\{\left.\frac{s}{s^2+b^2}\right|_{s\to s-1}\right\}\Big|_{t\to t-2}$$

$$= \text{step}(t-2)\left(e^t\cos bt\right)\Big|_{t\to t-2} = e^{t-2}\cos[b(t-2)]\,\text{step}(t-2)$$

(b)

$$\mathscr{L}\left\{(2-3t)\,\text{step}(t-1)\right\} = \mathscr{L}\left\{2\,\text{step}(t-1)\right\} - \mathscr{L}\left\{3t\,\text{step}(t-1)\right\} = \frac{2e^{-s}}{s} - 3e^{-s}\mathscr{L}\left\{t+1\right\}$$

$$= \frac{2e^{-s}}{s} - 3e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right) = -e^{-s}\left(\frac{1}{s}+\frac{3}{s^2}\right)$$

(c) $f(t) = 2 - 6\,\text{step}(t-3) + 5\,\text{step}(t-4)$

(d) $f(t) = t\,[\text{step}(t-1) - \text{step}(t-2)]$

■

5. [2360/042726 (15 pts)] Sid Phillips' pop tarts just came out of the toaster having a temperature of 50 °C. His mom places them in a room where the temperature is 20 °C when the wall clock says noon. At exactly 12:05 PM, Buzz Lightyear sees the pop tarts, is frightened and blasts them with his laser, giving them an impulsive temperature change of 6°C. Assuming Newton's law of cooling applies, this scenario is modeled by the differential equation $y' = -2(y-20) + 6\delta(t-5)$ where t is in minutes and $t = 0$ corresponds to 12:00 PM (noon). Using Laplace transforms, find the temperature of Sid's pop tarts at 12:04 PM and 12:06 PM. Zero credit for using any other method of solution.

SOLUTION:

$$\mathscr{L}\left\{y' + 2y = 40 + 6\delta(t-5)\right\}$$

$$sY(s) - y(0) + 2Y(s) = \frac{40}{s} + 6e^{-5s}$$

$$Y(s) = \frac{40}{s(s+2)} + \frac{6e^{-5s}}{s+2} + \frac{50}{s+2}$$

$$\frac{40}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} \quad \overset{\text{PFD}}{\longrightarrow} \quad \frac{20}{s} - \frac{20}{s+2}$$

$$Y(s) = \frac{20}{s} + \frac{30}{s+2} + \frac{6e^{-5s}}{s+2}$$

$$y(t) = \mathscr{L}^{-1}\left\{\frac{20}{s} + \frac{30}{s+2} + \frac{6e^{-5s}}{s+2}\right\} = 20 + 30e^{-2t} + 6e^{-2(t-5)}\text{step}(t-5)$$

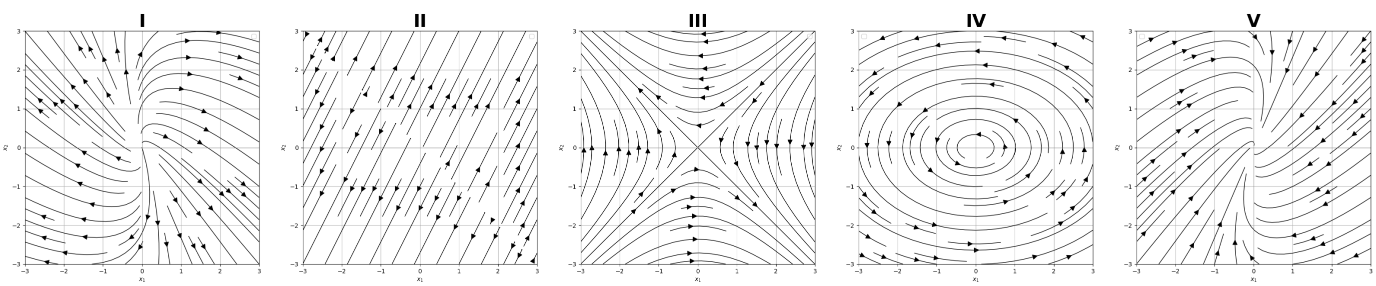
$$\text{At 12:04 PM}, t = 4 \implies y(4) = 20 + 30e^{-8}$$

$$\text{At 12:06 PM}, t = 6 \implies y(6) = 20 + 30e^{-12} + 6e^{-2}$$

■

6. [2360/042726 (21 pts)] Matrices, $\mathbf{A}$, in linear systems of the form $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ are given. For each part, classify the geometry and stability of the fixed point(s) and choose the correct phase portrait. Write NONE if no phase portrait matches. No work need be shown.

(a) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} -5 & 1 \\ -4 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} -2 & 1 \\ -4 & -2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ (f) $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ (g) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$



SOLUTION:

(a) Tr $\mathbf{A} = 3$; $|\mathbf{A}| = 0$; non-isolated fixed points, unstable, II

(b) Tr $\mathbf{A} = 0$; $|\mathbf{A}| = 2$; center, neutrally stable, IV

(c) Tr $\mathbf{A} = -6$; $|\mathbf{A}| = 9$; $\Delta = (\text{Tr}\mathbf{A})^2 - 4|\mathbf{A}| = 0$; (attracting) degenerate node, asymptotically stable, V

(d) Tr $\mathbf{A} = -4$; $|\mathbf{A}| = 8$; $\Delta = (\text{Tr}\mathbf{A})^2 - 4|\mathbf{A}| = -16$; (attracting) spiral, asymptotically stable, NONE

(e) Tr $\mathbf{A} = 3$; $|\mathbf{A}| = 2$; $\Delta = (\text{Tr}\mathbf{A})^2 - 4|\mathbf{A}| = 1$; (repelling) node, unstable, I

(f) Tr $\mathbf{A} = 8$; $|\mathbf{A}| = 16$; $\Delta = (\text{Tr}\mathbf{A})^2 - 4|\mathbf{A}| = 0$; (repelling) star node, unstable, NONE

(g) Tr $\mathbf{A} = 0$; $|\mathbf{A}| = -1$; saddle, unstable, III

■

7. [2360/042726 (18 pts)] Let $\mathbf{A} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) (3 pts) Show that the eigenvalues of $\mathbf{A}$ are $0, 1, 4$ by finding the roots of the characteristic equation.

(b) (15 pts) Use techniques shown in Homework12 to solve the variable coefficient initial value problem

$$t\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}, \; t > 0, \; \vec{\mathbf{x}}(1) = \begin{bmatrix} 6 \\ -2 \\ 5 \end{bmatrix}$$

writing your answer as a single vector. Use Cramer's Rule (zero credit for using a different method) to solve any linear system of algebraic equations that arise.

SOLUTION:

(a)

$$|\mathbf{A}-\lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(-1)^{3+3}\begin{vmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(2-\lambda)^2-4] = (1-\lambda)(4-4\lambda+\lambda^2-4)$$

$$= (1-\lambda)(\lambda^2-4\lambda) = \lambda(1-\lambda)(\lambda-4) = 0 \implies \lambda = 0, 1, 4$$

(b) We need to find the eigenvectors.

$$\lambda = 0 : \mathbf{A}\vec{\mathbf{v}}_0 = \vec{\mathbf{0}} : \left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}}_0 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda = 1 : (\mathbf{A}-\mathbf{I})\vec{\mathbf{v}}_1 = \vec{\mathbf{0}} : \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = 4 : (\mathbf{A}-4\mathbf{I})\vec{\mathbf{v}}_4 = \vec{\mathbf{0}} : \left[\begin{array}{ccc|c} -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

So the general solution is

$$\vec{\mathbf{x}}(t) = c_1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 t^4\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

to which we apply the initial condition, giving the linear system of algebraic equations

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 5 \end{bmatrix}$$

which is solved using Cramer's Rule as

$$c_1 = \frac{\begin{vmatrix} 6 & 0 & 1 \\ -2 & 0 & 1 \\ 5 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}} = \frac{-8}{2} = -4 \qquad c_2 = \frac{\begin{vmatrix} -1 & 6 & 1 \\ 1 & -2 & 1 \\ 0 & 5 & 0 \end{vmatrix}}{\begin{vmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}} = \frac{10}{2} = 5 \qquad c_3 = \frac{\begin{vmatrix} -1 & 0 & 6 \\ 1 & 0 & -2 \\ 0 & 1 & 5 \end{vmatrix}}{\begin{vmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}} = \frac{4}{2} = 2$$

The solution to the initial value problem is thus

$$\vec{\mathbf{x}}(t) = -4\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + 5t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 2t^4\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2t^4+4 \\ 2t^4-4 \\ 5t \end{bmatrix}$$

■

8. [2360/042726 (23 pts)] A horizontally oriented harmonic oscillator is governed by the equation $2\ddot{x}+8\dot{x}+26x=0$. The motion is started by displacing the mass 3 units to the right of the equilibrium position and releasing it from rest.

   (a) (2 pts) Give two reasons why the oscillator is not in resonance.

   (b) (4 pts) Show that the initial value problem is equivalent to the linear system $\vec{\mathbf{u}}' = \begin{bmatrix} 0 & 1 \\ -13 & -4 \end{bmatrix}\vec{\mathbf{u}}$, $\vec{\mathbf{u}}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ where $\vec{\mathbf{u}}$ is an unknown vector.

   (c) (15 pts) Solve the linear system from part (b). Zero credit for solving the original, given initial value problem. Hint: the final answer contains no fractions.

(d) (2 pts) Without computing any derivatives, what is the velocity of the mass when t is an integer multiple of π? Justify your answer.

SOLUTION:

(a) The oscillator is not in resonance since it is damped and unforced.

(b) Let $u_1 = x$ and $u_2 = \dot{x}$. Then $u_1' = \dot{x} = u_2$ and $u_2' = \ddot{x} = \frac{1}{2}(-26x - 8\dot{x}) = -13u_1 - 4u_2$. Furthermore $u_1(0) = x(0) = 3$ and $u_2(0) = \dot{x}(0) = 0$. Thus, with $\vec{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, we have

$$\vec{\mathbf{u}}' = \begin{bmatrix} 0 & 1 \\ -13 & -4 \end{bmatrix} \vec{\mathbf{u}}, \; \vec{\mathbf{u}}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

(c) Find the eigenvalues of the matrix.

$$\begin{vmatrix} -\lambda & 1 \\ -13 & -4-\lambda \end{vmatrix} = -\lambda(-4-\lambda) + 13 = \lambda^2 + 4\lambda + 13 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 4(1)(13)}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i$$

Find eigenvector corresponding to $\lambda = -2 + 3i$ by solving $[\mathbf{A} - (-2+3i)\mathbf{I}]\,\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

$$\left[\begin{array}{cc|c} 2-3i & 1 & 0 \\ -13 & -4-(-2+3i) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2-3i & 1 & 0 \\ -13 & -2-3i & 0 \end{array}\right] \xrightarrow{R_1^* = (2+3i)R_1}$$

$$\left[\begin{array}{cc|c} 13 & 2+3i & 0 \\ -13 & -2-3i & 0 \end{array}\right] \xrightarrow[R_1^* = \frac{1}{13}R_1]{R_2^* = 1R_1 + R_2} \left[\begin{array}{cc|c} 1 & \frac{2+3i}{13} & 0 \\ 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}} = \begin{bmatrix} -2-3i \\ 13 \end{bmatrix} = \begin{bmatrix} -2 \\ 13 \end{bmatrix} + i \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

$$\vec{\mathbf{u}}_{\text{real}} = e^{-2t}\left(\cos 3t \begin{bmatrix} -2 \\ 13 \end{bmatrix} - \sin 3t \begin{bmatrix} -3 \\ 0 \end{bmatrix}\right) = e^{-2t}\begin{bmatrix} -2\cos 3t + 3\sin 3t \\ 13\cos 3t \end{bmatrix}$$

$$\vec{\mathbf{u}}_{\text{imag}} = e^{-2t}\left(\sin 3t \begin{bmatrix} -2 \\ 13 \end{bmatrix} + \cos 3t \begin{bmatrix} -3 \\ 0 \end{bmatrix}\right) = e^{-2t}\begin{bmatrix} -2\sin 3t - 3\cos 3t \\ 13\sin 3t \end{bmatrix}$$

$$\vec{\mathbf{u}} = c_1 e^{-2t}\begin{bmatrix} -2\cos 3t + 3\sin 3t \\ 13\cos 3t \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} -2\sin 3t - 3\cos 3t \\ 13\sin 3t \end{bmatrix}$$

Now apply the initial condition to the above general solution to get

$$-2c_1 - 3c_2 = 3$$

$$13c_1 + 0c_2 = 0$$

giving $c_1 = 0$ and $c_2 = -1$. The solution to the initial value problem is thus

$$\vec{\mathbf{u}} = e^{-2t}\begin{bmatrix} 2\sin 3t + 3\cos 3t \\ -13\sin 3t \end{bmatrix}$$

(d) From the substitution, $u_2(t) = \dot{x}(t) = -13\sin 3t$. This is zero if t is an integer multiple of π meaning the velocity is zero at these values of t.

■