1. (34 pts) Determine whether each of the following series converge or diverge. On this exam, state the name of any test you use, and verify that the conditions for using that test are satisfied.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1+\ln(n)^2)}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 10^n}{n4^{2n+1}}$

   Solution:

   (a) These terms are always positive; applying the Limit Comparison Test with $\sum \frac{1}{n^2}$ gives

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n\sqrt{n^2-1}}} &= \lim_{n\to\infty} \frac{n\sqrt{n^2-1}}{n^2} \\
&= \lim_{n\to\infty} \frac{\sqrt{n^2-1}}{n} \\
&= \lim_{n\to\infty} \frac{n\sqrt{1-\frac{1}{n^2}}}{n} \\
&= \lim_{n\to\infty} \sqrt{1-\frac{1}{n^2}} \\
&= 1.
\end{aligned}$$

   (b) These terms are positive and decreasing; applying the Integral Test with $u = \ln(x)$ gives

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x(1+\ln(x)^2)}\,dx &= \int_0^{\infty} \frac{1}{1+u^2}\,du \\
&= \lim_{t\to\infty} \int_0^t \frac{1}{1+u^2}\,du \\
&= \lim_{t\to\infty} \arctan(t) \\
&= \frac{\pi}{2}.
\end{aligned}$$

   Since the integral converges, the series does as well.

   (c) Applyin the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{(-1)^{n+1}10^{n+1}}{(n+1)4^{2(n+1)+1}} \cdot \frac{n4^{2n+1}}{(-1)^n 10^n} \right| &= \lim_{n\to\infty} 10 \left(\frac{n}{n+1}\right) \frac{4^{2n+1}}{4^{2n+3}} \\
&= \lim_{n\to\infty} 10 \left(\frac{n}{n+1}\right) \frac{1}{4^2} \\
&= \frac{10}{16} \\
&= \frac{5}{8} < 1.
\end{aligned}$$

   Hence, the Ratio Test says that this series converges.

2. (14 pts) Find the center, radius of convergence, and interval of convergence for the series

$$\sum_{n=1}^{\infty} \frac{(3x+1)^{n+1}}{2\sqrt{n}+2}.$$

Solution: The center of this series is the x-value where $3x+1=0 \implies x=-1/3$. To find the radius and interval of convergence, we apply the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{(3x+1)^{n+2}}{2\sqrt{n+1}+2}\cdot\frac{2\sqrt{n}+2}{(3x+1)^{n+1}}\right| &= \lim_{n\to\infty}\left(\frac{2\sqrt{n}+2}{2\sqrt{n+1}+2}\right)|3x+1| \\
&= \lim_{n\to\infty}\left(\frac{\sqrt{n}+1}{\sqrt{n+1}}\right)|3x+1| \\
&= \lim_{n\to\infty}\left(\frac{\sqrt{n}}{\sqrt{n+1}}+\frac{1}{\sqrt{n+1}}\right)|3x+1| \\
&= |3x+1|
\end{aligned}$$

Setting $|3x+1|<1$ and solving for x gives the interval $-\frac{2}{3}<x<0$. The distance from the center to an endpoint is $R=1/3$.

Finally, we need to check the endpoints for convergence. Plugging in $x=-2/3$ yields the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2\sqrt{n}+2}.$$

These terms alternate in sign,

$$\lim_{n\to\infty}\frac{1}{2\sqrt{n}+2}=0,$$

and the positive part b_n is decreasing, since

$$b_n' = [(2\sqrt{n}+2)^{-1}]' = -\frac{1}{\sqrt{n}}(2\sqrt{n}+2)^{-2} < 0.$$

Hence, the Alternating Series test says that the series converges at $x=-2/3$.

At the other endpoint, we get the series

$$\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+2},$$

which diverges by a limit comparison with $\sum \frac{1}{\sqrt{n}}$:

$$\lim_{n\to\infty}\frac{\frac{1}{2\sqrt{n}+2}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\frac{\sqrt{n}}{2\sqrt{n}+2} \stackrel{LH}{=} \lim_{n\to\infty}\frac{1/(2\sqrt{n})}{1/\sqrt{n}} = \frac{1}{2}.$$

Therefore, the interval of convergence is $[-2/3, 0)$.

3. (16 pts) In the following two problems, use $f(x)=\dfrac{\sin(3x^2)}{x}$. You may assume that $f(0)=0$, which makes $f(x)$ continuous and differentiable.

   (a) Find a Maclaurin series representation of $f(x)$, and determine its radius of convergence.

   (b) Find the value of $f^{(9)}(0)$, meaning the value of the ninth derivative of $f(x)$ at $x=0$. You do not need to simplify your answer.

Solution:

(a) Using the given Maclaurin series for $\sin(x)$, we have

$$\begin{aligned}\frac{\sin(3x^2)}{x} &= \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n(3x^2)^{2n+1}}{(2n+1)!}\\ &= \sum_{n=0}\frac{(-1)^n 3^{2n+1}x^{4n+1}}{(2n+1)!}\end{aligned}$$

To find the radius of convergence R, we use the ratio test:

$$\begin{aligned}&\lim_{n\to\infty}\left|\frac{(-1)^{n+1}3^{2(n+1)+1}x^{4(n+1)+1}}{(2(n+1)+1)!}\frac{(2n+1)!}{(-1)^n 3^{2n+1}x^{4n+1}}\right|\\ &= \lim_{n\to\infty}\left|\frac{3^{2n+3}x^{4n+5}}{(2n+3)!}\frac{(2n+1)!}{3^{2n+1}x^{4n+1}}\right|\\ &= \lim_{n\to\infty}\left|\frac{3^2x^4}{(2n+3)(2n+2)}\right|\\ &= 0\end{aligned}$$

The limit equals 0 for all values of x so we have $R=\infty$

(b) The $f^{(9)}$ value occurs as part of the coefficient of the x^9 term, according to Taylor's Theorem. In our series above, the x^9 term corresponds to when $4n+1=9$, so when $n=2$. Using the formula for coefficients of a Taylor series:

$$\frac{f^{(9)}(0)}{9!} = \frac{(-1)^2 3^{2(2)+1}}{(2(2)+1)!} = \frac{3^5}{5!}$$

So we have

$$f^{(9)}(0) = \frac{9!}{5!}3^5.$$

4. (22 pts) The following problems are not related.

(a) Consider the function $h'(x) = \dfrac{x}{1+9x^2}$.

i. Find a power series representation for $h'(x)$.

ii. Use the power series from part (i) to find a power series representation for $g(x) = 4\cdot h(x)$, assuming that $h(0)=1$.

(b) Determine whether $\displaystyle\sum_{n=0}^{\infty}\frac{-2}{(n+1)(n+2)}$ converges or diverges. If it converges, find its sum.

Solution:

(a) (i) Using the sum formula for geometric series:

$$h'(x) = \frac{x}{1+9x^2} = x\sum_{n=0}^{\infty}(-9x^2)^n = \sum_{n=0}^{\infty}(-9)^n x^{2n+1}.$$

(ii)

$$h(x) = \int\sum_{n=0}^{\infty}(-9)^n x^{2n+1}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} (-9)^n \int x^{2n+1}\, dx \\
&= C + \sum_{n=0}^{\infty} \frac{(-9)^n x^{2n+2}}{2n+2}
\end{aligned}
$$

This implies that $1 = h(0) = C$. Thus,

$$g(x) = 4h(x) = 4 + 4\sum_{n=0}^{\infty} \frac{(-9)^n x^{2n+2}}{2n+2}$$

(b) Using partial fraction decomposition,

$$\frac{-2}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}.$$

Clearing denominators gives

$$-2 = A(n+2) + B(n+1),$$

And plugging in $n = -2$, $n = -1$ gives $A = -2$, $B = 2$. Hence,

$$\frac{-2}{(n+1)(n+2)} = \frac{-2}{n+1} + \frac{2}{n+2}.$$

This means that partial sums of the series look like

$$s_n = \left(\frac{-2}{1} + \frac{2}{2}\right) + \left(-\frac{2}{2} + \frac{2}{3}\right) + \cdots + \left(-\frac{2}{n+1} + \frac{2}{n+2}\right) = -2 + \frac{2}{n+2}.$$

Sending $n \to \infty$ shows that the series converges to -2.

5. (14 pts) The following two problems are related:

(a) Show that $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} e^{-2n+1}$ converges.

(b) How many terms of the series in (a) are needed to guarantee a sum within 0.01 of the true value? You should solve for n, but you do not need to simplify any numerical expression.

Solution:

(a) This is an alternating series, and the terms have positive part $b_n = e^{-2n+1}$. Note that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{e}{e^{2n}} = 0$$

and

$$b_n' = -2e^{-2n+1} < 0,$$

since $e^x > 0$ for all x. Hence, the sequence b_n is decreasing. By the Alternating Series Test, this series converges.

(b) The Alternating Series Estimation Theorem says that

$$|s - s_n| \le b_{n+1}.$$

Using this fact, we can guarantee $|s - s_n| < 0.01$ if we choose n so that

$$b_{n+1} = e^{-2(n+1)+1} = e^{-2n-1} < 0.01.$$

Solving for n, we get

$$-2n - 1 < \ln(0.01) \implies n > -\frac{1}{2}(\ln(0.01) + 1).$$

Alternatively, we can use the estimate that $2 < e$, so

$$\frac{1}{e^{2n+1}} < \frac{1}{2^{2n+1}}.$$

Noting that $2^7 = 128 > 100$, and $2n + 1 = 7$ implies that $n = 3$, we conclude that at least three terms of the series will give a sum within 0.01 of the true value.