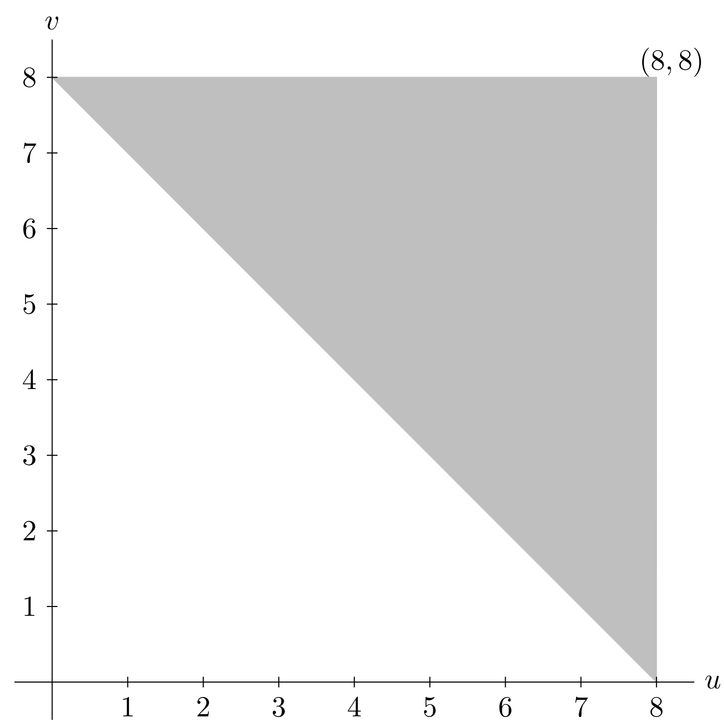
1. Consider the polygonal region in the xy-plane, R_{xy}, with vertices $(2,-1)$, $(2,3)$, and $(4,2)$. We will consider $\iint_{R_{xy}} 32x\,dA$ using the substitution $u = 3x - 2y$ and $v = x + 2y$.

   (a) (6 points) Transform the original region R_{xy} into its corresponding region R_{uv} in the uv-plane. Make a clear sketch, of the new region of integration R_{uv} in the uv-plane. Be sure to label all axes and intersection points on your sketch, and to shade in the region.

   (b) (13 points) Rewrite $\iint_{R_{xy}} 32x\,dA$ over the region R_{uv} in the uv-plane in terms of u and v. You do **NOT** need to evaluate this integral.

Solution:

(a) We have the following sketch:



(b) We need to find x and y in terms of u and v. Note that if we add the given equations, we obtain $u + v = 4x$, which yields

$$x = \frac{u}{4} + \frac{v}{4}.$$

If we plug this back into one of the original equations and solve for y, we obtain

$$y = -\frac{1}{8}u + \frac{3}{8}v.$$

The determinant of the Jacobian is the absolute value of

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \frac{1}{4}\cdot\frac{3}{8} - \frac{1}{4}\left(-\frac{1}{8}\right) = \frac{1}{8}.$$

Using our sketch from (b) for the limits of integration, the integral becomes

$$\iint_{R_{xy}} 32x\,dA = \int_0^8 \int_{8-u}^8 32\left(\frac{1}{4}u + \frac{1}{4}v\right)\left|\frac{1}{8}\right| dv\,du = \int_0^8 \int_{8-u}^8 u + v\,dv\,du$$

2. Consider the integral

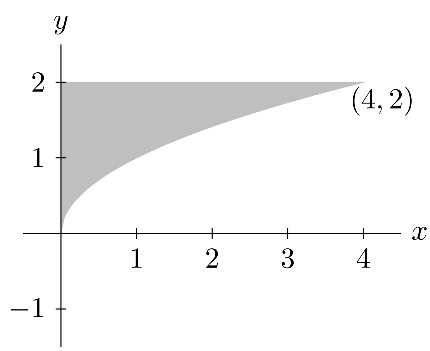
$$\int_0^4 \int_{\sqrt{x}}^2 \frac{4x}{y^5+1}\,dy\,dx.$$

 (a) (6 points) Sketch the region over which we are integrating in the xy-plane. Be sure to label all axes and intersection points on your sketch, and to shade in the region.

 (b) (13 points) Evaluate the integral.

Solution:

 (a) We have the following sketch:



 (b) We switch the order of integration to obtain:

$$\begin{aligned}
\int_0^2 \int_0^{y^2} \frac{4x}{y^5+1}\,dx\,dy &= \int_0^2 \left[\frac{2x^2}{y^5+1}\right]_0^{y^2} dy \\
&= \int_0^2 \frac{2y^4}{y^5+1}\,dy \\
&= \frac{2}{5}\int_1^{33} \frac{1}{u}\,du \qquad \text{(via the substitution } u = y^5+1\text{)} \\
&= \frac{2\ln 33}{5}.
\end{aligned}$$

3. Poppy the Pudu is baking a blueberry pie for her friend Sam the Squirrel. The pie occupies the region bounded **below** by the plane $z = 3$, bounded to the **side** by the cone $x^2 + y^2 = z^2$, and bounded **above** by the sphere $x^2 + y^2 + z^2 = 25$.

 (a) (6 points) Sketch a cross section of this pie on the rz-plane. Be sure to label all axes and intersection points on your sketch, and to shade in the region.

 (b) (18 points) The density of blueberries in this pie is

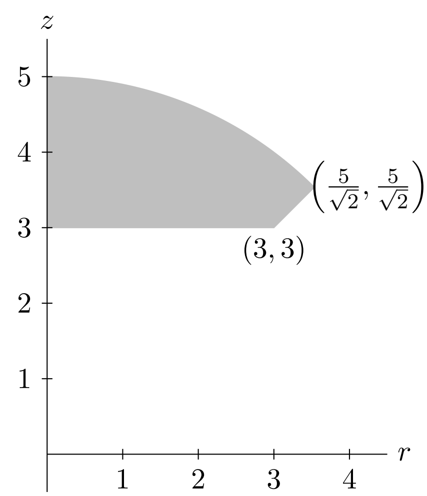
$$b(x, y, z) = x^2 + y^2 + z^2 \text{ blueberries per cubic inch.}$$

 Set up **but do not evaluate** the integrals or sum of integrals that give the total number of blueberries in this pie in two ways:

i. Using the integration $dr\,dz\,d\theta$.
ii. Using the integration $d\rho\,d\phi\,d\theta$.

Solution:

(a) We have the following region:



(b) i.

$$\int_0^{2\pi}\int_3^{\frac{5}{\sqrt{2}}}\int_0^{z}(r^2+z^2)r\,dr\,dz\,d\theta+\int_0^{2\pi}\int_{\frac{5}{\sqrt{2}}}^{5}\int_0^{\sqrt{25-z^2}}(r^2+z^2)r\,dr\,dz\,d\theta$$

ii.

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{3\sec\phi}^{5}\rho^4\sin\phi\,d\rho\,d\phi\,d\theta$$

4. (19 points) Pam the Penguin finds the previously mentioned pie cooling on the counter and decides to pipe some icing onto the pie along the curve that is the intersection of the surfaces

$$x^2+y^2+z^2=25 \qquad \text{and} \qquad x^2+y^2=9$$

where $z \geq 0$. The density of this icing is given by

$$c(x,y,z)=x^2z \text{ grams per inch.}$$

Find the total mass of the icing that was added to the top of the pie.

Solution:

At the intersection, we have $z^2=16$, so $z=4$, where $x^2+y^2=9$. So one way to parameterize the curve is by

$$\mathbf{r}(t)=\langle 3\cos t, 3\sin t, 4\rangle, \qquad 0\leq t\leq 2\pi.$$

We have

$$\mathbf{r}'(t)=\langle -3\sin t, 3\cos t, 0\rangle,$$

so

$$||\mathbf{r}'(t)||=\sqrt{9\sin^2 t+9\cos^2 t+0^2}=3.$$

Then, we have

$$
\begin{aligned}
\int_C x^2 z\, ds &= \int_0^{2\pi} f(\mathbf{r}(t))||\mathbf{r}'(t)||\, dt \\
&= \int_0^{2\pi} (3\cos t)^2 (4)(3)\, dt \\
&= 108 \int_0^{2\pi} \cos^2 t\, dt \\
&= 54 \int_0^{2\pi} 1 + \cos(2t)\, dt \\
&= 54 \left[t - \frac{1}{2}\sin(2t) \right]_0^{2\pi} \\
&= 108\pi \text{ grams.}
\end{aligned}
$$

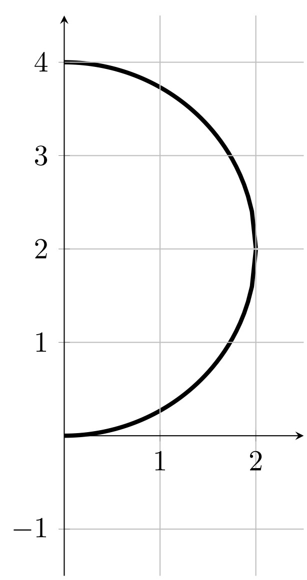
5. (19 points) Not to be outdone, Sam the Squirrel decides to bake his famous chocolate frosted cupcakes for Poppy the Pudu and Pam the Penguin. He only puts chocolate frosting on the top of a cupcake. The top of the cupcake's surface is given by

$$x^2 + y^2 + (z-2)^2 = 4, \qquad z \geq 3.$$

where the units of x, y, and z are inches. Use Calculus 3 techniques to find the surface area of the top of one of these cupcakes.

Solution:

In the rz-plane, the sphere looks like the following:



We will parameterize the surface of the top of a cupcake with r and θ. Note that the surface can be rewritten in terms of r and z, and then we can get z in terms of r:

$$
\begin{aligned}
x^2 + y^2 + (z-2)^2 &= 4 \\
r^2 + (z-2)^2 &= 4 \\
(z-2)^2 &= 4 - r^2
\end{aligned}
$$

$$\begin{aligned} z-2 &= \sqrt{4-r^2} \qquad \text{(recall } z \geq 3\text{, so } z-2 \geq 0) \\ z &= 2+\sqrt{4-r^2}. \end{aligned}$$

So, we can use the parameterization given by

$$\mathbf{r}(r,\theta) = \left\langle r\cos\theta, r\sin\theta, 2+\sqrt{4-r^2}\right\rangle.$$

We have

$$\begin{aligned} \mathbf{r}_r &= \left\langle \cos\theta, \sin\theta, -\frac{r}{\sqrt{4-r^2}}\right\rangle, \\ \mathbf{r}_\theta &= \langle -r\sin\theta, r\cos\theta, 0\rangle, \\ \mathbf{r}_r \times \mathbf{r}_\theta &= \left\langle -\frac{r^2\cos\theta}{\sqrt{4-r^2}}, -\frac{r^2\sin\theta}{\sqrt{4-r^2}}, r\right\rangle, \qquad \text{and} \\ ||\mathbf{r}_r \times \mathbf{r}_\theta|| &= \sqrt{\frac{r^4\cos^2\theta}{4-r^2}+\frac{r^4\sin^2\theta}{4-r^2}+r^2} = \frac{2r}{\sqrt{4-r^2}}. \end{aligned}$$

It's clear we have $0 \leq \theta \leq 2\pi$. To determine the upper limit of r, we plug $z=3$ into $r^2+(z-2)^2=4$ and solve for r to obtain $r=\sqrt{3}$. So, we have $0 \leq r \leq \sqrt{3}$.

Thus, the surface area is given by

$$\begin{aligned} \int_0^{\sqrt{3}}\int_0^{2\pi} \frac{2r}{\sqrt{4-r^2}}\,d\theta\,dr &= 2\pi\int_0^{\sqrt{3}} \frac{2r}{\sqrt{4-r^2}}\,dr \\ &= -2\pi\int_4^1 u^{-1/2}\,du \qquad \text{(where } u=4-r^2) \\ &= -4\pi\, u^{1/2}\Big|_4^1 \\ &= 4\pi \text{ square inches.} \end{aligned}$$