1. [2360/040826 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a two-column table (letter - answer) completely separate from any work you do to arrive at the answer.

   (a) The differential equation $y^{(4)} - y''' = t^2$ is equivalent to the system of differential equations given by

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1+t^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

   (b) $\displaystyle\int_0^\infty t\cos t e^{-st}\,\mathrm{d}t = \frac{s^2-1}{(s^2+1)^2}$.

   (c) The total energy, E, of the system governed by $2\ddot{x} + e^{-x} = -1$ is $E = \dot{x}^2 + x - e^{-x}$.

   (d) An harmonic oscillator with $m = b = k = 1$ and forced by $\cos 1.01t$ will exhibit beats.

   (e) $\mathscr{L}\left\{e^{-t}\cosh t\right\} = \dfrac{s+1}{s^2+2s}$.

SOLUTION:

   (a) **FALSE** The correct system is

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{bmatrix}$$

   (b) **TRUE**

$$\int_0^\infty t\cos t e^{-st}\,\mathrm{d}t = \mathscr{L}\{t\cos t\} = -\frac{\mathrm{d}}{\mathrm{d}s}\mathscr{L}\{\cos t\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{s^2+1}\right) = \frac{s^2-1}{(s^2+1)^2}$$

   (c) **TRUE** Since $m = 2$, the kinetic energy is $\dot{x}^2$. Rewriting the equation as $2\ddot{x} + (e^{-x}+1) = 0$ shows that $V'(x) = e^{-x} + 1$ so that $V(x) = x - e^{-x}$ giving the total energy as $E = \dot{x}^2 + x - e^{-x}$.

   (d) **FALSE** Only undamped oscillators can exhibit beats.

   (e) **TRUE**

$$\mathscr{L}\left\{e^{-t}\cosh t\right\} = \mathscr{L}\{\cosh t\}\Big|_{s\to s+1} = \frac{s}{s^2-1}\Big|_{s\to s+1} = \frac{s+1}{(s+1)^2-1} = \frac{s+1}{s^2+2s}$$

■

2. [2360/040826 (22 pts)] Consider the linear operator $\mathcal{L}(\vec{\mathbf{y}}) = y'''$.

   (a) (3 pts) Use an appropriate characteristic equation to find a basis for the solution space of $\mathcal{L}(\vec{\mathbf{y}}) = 0$ and state the dimension of the solution space.

   (b) (6 pts) Is $\left\{3t^2 - 7t + \frac{1}{2}, t+1, 2t^2 - 11t - 6\right\}$ a basis for the solution space of $\mathcal{L}(\vec{\mathbf{y}}) = 0$? Justify your answer.

   (c) (13 pts) Solve the initial value problem $\mathcal{L}(\vec{\mathbf{y}}) = 120(t^3 + t - 2)$, $y(0) = 1$, $y'(0) = 2$, $y''(0) = 6$ using the Method of Undetermined Coefficients. Zero credit for using any other technique.

SOLUTION:

   (a) The characteristic equation is $r^3 = 0$ which has the root 0 with multiplicity 3. A basis for the solution space is $\left\{e^{0t}, te^{0t}, t^2e^{0t}\right\}$ or $\left\{1, t, t^2\right\}$. The solution space has dimension 3.

   (b) The dimension of the solution space is 3 so the set contains the right number of functions.

   - Check that the functions are in the solution space:

$$\left(3t^2 - 7t + \frac{1}{2}\right)''' = 0 + 0 + 0 = 0$$

$$(t+1)''' = 0 + 0 + 0 = 0$$

$$\left(2t^2 - 11t - 6\right)''' = 0 + 0 + 0 = 0$$

   Thus, all functions are solutions to the differential equation and thus in the solution space.

- Check for linear independence:

$$W(t)=\begin{vmatrix}3t^2-7t+\frac{1}{2} & t+1 & 2t^2-11t-6\\ 6t-7 & 1 & 4t-11\\ 6 & 0 & 4\end{vmatrix}=6(-1)^{3+1}\begin{vmatrix}t+1 & 2t^2-11t-6\\ 1 & 4t-11\end{vmatrix}+4(-1)^{3+3}\begin{vmatrix}3t^2-7t+\frac{1}{2} & t+1\\ 6t-7 & 1\end{vmatrix}$$

$$=6\left[4t^2-11t+4t-11-\left(2t^2-11t-6\right)\right]+4\left[3t^2-7t+\frac{1}{2}-\left(6t^2+6t-7t-7\right)\right]$$

$$=6\left(2t^2+4t-5\right)+4\left(-3t^2-6t+\frac{15}{2}\right)=12t^2+24t-30-12t^2-24t+30=0$$

Since the Wronskian vanishes, the functions are linearly dependent and thus cannot form a basis for the solution space.

Alternatively, consider $c_1(3t^2-7t+\frac{1}{2})+c_2(t+1)+c_3(2t^2-11t-6)=0t^2+0t+0$. Equating coefficients on both sides of the equation yields the following system

$$\left.\begin{aligned}3c_1+2c_3&=0\\ -7c_1+c_2-11c_3&=0\\ \tfrac{1}{2}c_1+c_2-6c_3&=0\end{aligned}\right\}\implies\begin{bmatrix}3&0&2\\-7&1&-11\\ \frac{1}{2}&1&-6\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{vmatrix}3&0&2\\-7&1&-11\\ \frac{1}{2}&1&-6\end{vmatrix}=1(-1)^{2+2}\begin{vmatrix}3&2\\ \frac{1}{2}&-6\end{vmatrix}+1(-1)^{3+2}\begin{vmatrix}3&2\\-7&-11\end{vmatrix}=-19-(-19)=0$$

indicating the system has nontrivial solutions and therefore the functions are linearly dependent.

(c) The initial guess for the particular solution is $y_p=At^3+Bt^2+Ct+D$ which must be modified to $y_p=At^6+Bt^5+Ct^4+Dt^3$. Substituting into the DE,

$$y_p'''=120At^3+60Bt^2+24Ct+6D=120t^3+120t-240$$

$$t^3:120A=120\implies A=1$$

$$t^2:60B=0\implies B=0$$

$$t^1:24C=120\implies C=5$$

$$t^0:6D=-240\implies D=-40$$

Thus, $y_p=t^6+5t^4-40t^3$ and $y=y_h+y_p=c_1+c_2t+c_3t^2-40t^3+5t^4+t^6$. Now apply the initial conditions.

$$y(0)=c_1=1$$

$$y'(t)=c_2+2c_3t-120t^2+25t^4+6t^5\implies y'(0)=c_2=2$$

$$y''(t)=2c_3-240t+100t^3+30t^4\implies y''(0)=2c_3=6\implies c_3=3$$

Finally, the solution to the initial value problem is $y(t)=1+2t+3t^2-40t^3+5t^4+t^6$

■

3. [2360/040826 (20 pts)] Use variation of parameters to solve the initial value problem

$$x^2y''+2xy'-2y=30x^3+30x,\ y(1)=4\,,y'(1)=23$$

Assume $x>0$. Zero credit for using any other technique. Hint: Your final answer will not contain fractions.

SOLUTION:

This is a Cauchy-Euler equation so assume solutions of the form $y=x^r$ and solve the associated homogeneous problem.

$$x^2y''+2xy'-2y=x^2r(r-1)x^{r-2}+2xrx^{r-1}-2x^r=x^r\left(r^2-r+2r-2\right)=0$$

$$\implies r^2+r-2=0\implies(r+2)(r-1)=0\implies r=-2,1$$

So let $y_1 = x, y_2 = x^{-2}$ and $y_p = v_1 y_1 + v_2 y_2$. The differential equation in the correct form for variation of parameters yields a right hand side of $f(x) = 30x + 30/x$.

$$W(x) = \begin{vmatrix} x & x^{-2} \\ 1 & -2x^{-3} \end{vmatrix} = -3x^{-2}$$

$$v_1 = \int \frac{-x^{-2}\left(30x + 30/x\right)}{-3x^{-2}}\,\mathrm{d}x = 10\int \left(x + \frac{1}{x}\right)\mathrm{d}x = 5x^2 + 10\ln x \quad (x > 0)$$

$$v_2 = \int \frac{x\left(30x + 30/x\right)}{-3x^{-2}}\,\mathrm{d}x = -10\int \left(x^4 + x^2\right)\mathrm{d}x = -2x^5 - \frac{10}{3}x^3$$

$$y_p = \left(5x^2 + 10\ln x\right)x + \left(-2x^5 - \frac{10}{3}x^3\right)x^{-2} = 3x^3 + 10x\ln x - \frac{10}{3}x$$

$$y = y_h + y_p = c_1 x + c_2 x^{-2} + 3x^3 + 10x\ln x - \frac{10}{3}x = c_1 x + c_2 x^{-2} + 3x^3 + 10x\ln x \qquad \text{after absorbing } -\frac{10}{3} \text{ into } c_1$$

Now apply the initial conditions

$$y(1) = c_1 + c_2 + 3 = 4 \implies c_1 + c_2 = 1$$

$$y'(x) = c_1 - 2c_2 x^{-3} + 9x^2 + 10 + 10\ln x \implies y'(1) = c_1 - 2c_2 + 19 = 23 \implies c_1 - 2c_2 = 4$$

Using Cramer's Rule

$$c_1 = \frac{\begin{vmatrix} 1 & 1 \\ 4 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{-6}{-3} = 2 \qquad c_2 = \frac{\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{3}{-3} = -1$$

Finally, $y(x) = 2x - x^{-2} + 3x^3 + 10x\ln x$. Note that had the $-\frac{10}{3}$ not been absorbed into c_1, the resultant system would have been

$$c_1 + c_2 = \frac{13}{3}$$
$$c_1 - 2c_2 = \frac{22}{3}$$

which has $c_1 = \frac{16}{3}, c_2 = -1$ as solution, yielding the same solution to the IVP. ■

4. [2360/040826 (10 pts)] For each of the following, a characteristic equation and forcing function, $f(t)$, or a nonhomogeneous differential equation is given. Provide the guess for the particular solution you would use when implementing the Method of Undetermined Coefficients. **DO NOT** find the coefficients, simply write your guess. If the method is not applicable, write **N/A**.

(a) $r^3 + r^2 - 2 = 0, f(t) = e^t(1 + \cos t)$

(b) $y'' + 2y' + 3y = \ln t$

(c) $r(r^2 + 4)^2 = 0, f(t) = 6 + 7\sin 3t - 5\cos 3t$

(d) $y'' + 16y = e^{2t}\cos 4t + \sin 2t$

(e) $(r - 1)^2(r^2 + 3r + 2) = 0, f(t) = e^{-3t} + t(e^t + 2)$

SOLUTION:

(a) The potential rational roots are $\pm 1, \pm 2$. Synthetic division gives

$$\begin{array}{c|cccc} & 1 & 1 & 0 & -2 \\ 1 & & 1 & 2 & 2 \\ \hline & 1 & 2 & 2 & 0 \end{array}$$

implying that $r = 1$ is a root. For the resulting quadratic, $r^2 + 2r + 2 = 0$ we have

$$r = \frac{-2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i$$

Alternatively, by inspection $r = 1$ is a root meaning that $r - 1$ is a factor. Polynomial long division then gives the same quadratic as before.

$$
\begin{array}{r}
r^2 + 2r + 2 \\
r - 1 \overline{\smash{\big)}\; r^3 \;\; + r^2 \qquad - 2} \\
\underline{- r^3 \;\; + r^2} \qquad\qquad \\
2r^2 \qquad\quad \\
\underline{- 2r^2 + 2r} \quad \\
2r - 2 \\
\underline{- 2r + 2} \\
0
\end{array}
$$

Basis of the solution space for the homogeneous equation: $\{e^t, e^{-t}\cos t, e^{-t}\sin t\}$; $y_p = e^t(At + B\cos t + C\sin t)$

(b) N/A

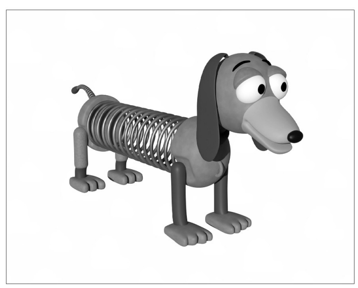
(c) $r = 0; r = \pm 2i$ with multiplicity 2; basis of the solution space for the homogeneous equation: $\{1, \cos 2t, \sin 2t, t\cos 2t, t\sin 2t\}$; $y_p = At + B\cos 3t + C\sin 3t$

(d) $r^2 + 16 = 0 \implies r = \pm 4i$; basis of the solution space for the homogeneous equation: $\{\cos 4t, \sin 4t\}$; $y_p = e^{2t}(A\cos 4t + B\sin 4t) + C\cos 2t + D\cos 2t$

(e) $(r-1)^2(r^2+3r+2) = (r-1)^2(r+1)(r+2) = 0 \implies r = -1, -2$ and $r = 1$ with multiplicity 2; basis of the solution space for the homogeneous equation: $\left\{e^t, te^t, e^{-t}, e^{-2t}\right\}$; $y_p = At + B + (Ct^3 + Dt^2)e^t + Ee^{-3t}$

$\blacksquare$

5. [2360/040826 (38 pts)] Andy's Slinky dog (Slink) can be considered a damped harmonic oscillator (see figure). Consider Slink's head to be the 2-kg mass in the oscillator, shown in the equilibrium (rest) position. The spring/restoring constant is 8 N/m. Unfortunately, Slink was coming into the house and his backside got caught in the doggie door. Not to worry - Buzz Lightyear to the rescue!



(a) (2 pts) How far to the right of the equilibrium position would Buzz stretch Slink if he exerts a force of 16 N? Include units in your answer.

(b) (4 pts) If Buzz wants Slink's head to pass through the equilibrium position more than once, find all values of the damping constant that make this possible.

(c) (27 pts) Now suppose that the damping constant $b = 10$ N/m/s and the mass and restoring constant remain 2 kg and 8 N/m, respectively. Buzz pulls Slink's head 2 m to the right of the equilibrium position and then gives it an initial velocity of 11 m/s to the left.

  i. (1 pt) In this case, is Slink underdamped, overdamped or critically damped?

  ii. (20 pts) Use Laplace transforms to find the equation of motion of Slink's head assuming there are no external forces acting on Slink. Zero credit for using any other technique.

  iii. (6 pts) Will Slink's head pass through the equilibrium position? If so, find the time(s) when this occurs. If not, explain why not.

(d) (5 pts) If all else remains the same, but an external force of $f(t) = 30\cos 2t$ is applied to Slink, the resulting Laplace transform of the initial value problem is

$$X(s) = \frac{3}{s^2+4} - \frac{2}{s+1} + \frac{4}{s+4}$$

Find the amplitude of the steady state motion of Slink's head in this case.

SOLUTION:

(a) Since Buzz stretches Slink, the restoring force is negative:

$$F = -kx \implies x = -F/k = -\frac{-16 \text{ N}}{8 \text{ N/m}} = 2 \text{ m}$$

(b) We need to find the damping constant, $b > 0$, that makes Slink underdamped, that is $b^2 - 4mk < 0$.

$$b^2 - 4mk = b^2 - 4(2)(8) = b^2 - 64 < 0 \implies b^2 < 64 \implies |b| < 8 \implies 0 < b < 8$$

(c) i. $b^2 - 4mk = 10^2 - 4(2)(8) = 100 - 64 = 36 > 0 \implies$ overdamped

ii. With $x(t)$ the displacement of Slink's head from the equilibrium position, we need to solve the initial value problem $2\ddot{x} + 10\dot{x} + 8x = 0,\ x(0) = 2,\ \dot{x}(0) = -11.$

$$\mathscr{L}\left\{2\ddot{x} + 10\dot{x} + 8\dot{x} = 0\right\}$$

$$2\left[s^2X(s) - sx(0) - \dot{x}(0)\right] + 10\left[sX(s) - x(0)\right] + 8X(s) = 0$$

$$\left(2s^2 + 10s + 8\right)X(s) = 2sx(0) + 2\dot{x}(0) + 10x(0)$$

$$X(s) = \frac{4s - 22 + 20}{2s^2 + 10s + 8} = \frac{2(2s-1)}{2(s^2 + 5s + 4)} = \frac{2s-1}{s^2 + 5s + 4} = \frac{2s-1}{(s+1)(s+4)}$$

$$\frac{2s-1}{(s+1)(s+4)} = \frac{A}{s+1} + \frac{B}{s+4}$$

$$2s - 1 = A(s+4) + B(s+1)$$

$$s = -1 : -3 = A(3) \implies A = -1$$

$$s = -4 : -9 = B(-3) \implies B = 3$$

$$x(t) = \mathscr{L}^{-1}\left\{\frac{2s-1}{(s+1)(s+4)}\right\} = \mathscr{L}^{-1}\left\{\frac{-1}{s+1} + \frac{3}{s+4}\right\} = -e^{-t} + 3e^{-4t}$$

iii. We need to see if a t exists such that $0 = -e^{-t} + 3e^{-4t}$.

$$\left[e^{-t} = 3e^{-4t}\right]e^t$$

$$\frac{1}{3} = e^{-3t}$$

$$\ln\frac{1}{3} = -3t$$

$$\frac{1}{3}\ln 3 = t$$

Slink's head passes through the equilibrium position when $t = \dfrac{1}{3}\ln 3$ sec.

(d)

$$x(t) = \mathscr{L}^{-1}\left\{X(s)\right\} = \mathscr{L}^{-1}\left\{\frac{3}{s^2+4} - \frac{2}{s+1} + \frac{4}{s+4}\right\} = \frac{3}{2}\sin 2t - 2e^{-t} + 4e^{-4t}$$

The steady state portion of the solution is $\frac{3}{2}\sin 2t$, the amplitude of which is $\frac{3}{2}$.

■