- This exam is worth 100 points and has 5 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"× 11" crib sheet with writing on one side.

0. At the top of the page containing your solution to problem 1, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/040826 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a two-column table (letter - answer) completely separate from any work you do to arrive at the answer.

   (a) The differential equation $y^{(4)} - y''' = t^2$ is equivalent to the system of differential equations given by

   $$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1+t^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

   (b) $\displaystyle\int_0^\infty t \cos t e^{-st}\,\mathrm{d}t = \frac{s^2-1}{(s^2+1)^2}$.

   (c) The total energy, E, of the system governed by $2\ddot{x} + e^{-x} = -1$ is $E = \dot{x}^2 + x - e^{-x}$.

   (d) An harmonic oscillator with $m = b = k = 1$ and forced by $\cos 1.01t$ will exhibit beats.

   (e) $\mathscr{L}\left\{e^{-t}\cosh t\right\} = \dfrac{s+1}{s^2+2s}$.

2. [2360/040826 (22 pts)] Consider the linear operator $\mathcal{L}(\vec{\mathbf{y}}) = y'''$.

   (a) (3 pts) Use an appropriate characteristic equation to find a basis for the solution space of $\mathcal{L}(\vec{\mathbf{y}}) = 0$ and state the dimension of the solution space.

   (b) (6 pts) Is $\left\{3t^2 - 7t + \frac{1}{2}, t+1, 2t^2 - 11t - 6\right\}$ a basis for the solution space of $\mathcal{L}(\vec{\mathbf{y}}) = 0$? Justify your answer.

   (c) (13 pts) Solve the initial value problem $\mathcal{L}(\vec{\mathbf{y}}) = 120(t^3 + t - 2),\ y(0) = 1,\ y'(0) = 2,\ y''(0) = 6$ using the Method of Undetermined Coefficients. Zero credit for using any other technique.

3. [2360/040826 (20 pts)] Use variation of parameters to solve the initial value problem

   $$x^2y'' + 2xy' - 2y = 30x^3 + 30x,\ y(1) = 4\ , y'(1) = 23$$

   Assume $x > 0$. Zero credit for using any other technique. Hint: Your final answer will not contain fractions.

4. [2360/040826 (10 pts)] For each of the following, a characteristic equation and forcing function, $f(t)$, or a nonhomogeneous differential equation is given. Provide the guess for the particular solution you would use when implementing the Method of Undetermined Coefficients. **DO NOT** find the coefficients, simply write your guess. If the method is not applicable, write **N/A**.

   (a) $r^3 + r^2 - 2 = 0, f(t) = e^t(1 + \cos t)$

   (b) $y'' + 2y' + 3y = \ln t$

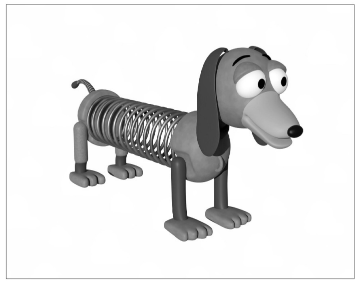
   (c) $r(r^2+4)^2 = 0, f(t) = 6 + 7\sin 3t - 5\cos 3t$

   (d) $y'' + 16y = e^{2t}\cos 4t + \sin 2t$

   (e) $(r-1)^2(r^2+3r+2) = 0, f(t) = e^{-3t} + t(e^t + 2)$

MORE PROBLEMS AND LAPLACE TRANSFORM TABLE BELOW/ON REVERSE

5. [2360/040826 (38 pts)] Andy's Slinky dog (Slink) can be considered a damped harmonic oscillator (see figure). Consider Slink's head to be the 2-kg mass in the oscillator, shown in the equilibrium (rest) position. The spring/restoring constant is 8 N/m. Unfortunately, Slink was coming into the house and his backside got caught in the doggie door. Not to worry - Buzz Lightyear to the rescue!



(a) (2 pts) How far to the right of the equilibrium position would Buzz stretch Slink if he exerts a force of 16 N? Include units in your answer.

(b) (4 pts) If Buzz wants Slink's head to pass through the equilibrium position more than once, find all values of the damping constant that make this possible.

(c) (27 pts) Now suppose that the damping constant $b = 10$ N/m/s and the mass and restoring constant remain 2 kg and 8 N/m, respectively. Buzz pulls Slink's head 2 m to the right of the equilibrium position and then gives it an initial velocity of 11 m/s to the left.

i. (1 pt) In this case, is Slink underdamped, overdamped or critically damped?

ii. (20 pts) Use Laplace transforms to find the equation of motion of Slink's head assuming there are no external forces acting on Slink. Zero credit for using any other technique.

iii. (6 pts) Will Slink's head pass through the equilibrium position? If so, find the time(s) when this occurs. If not, explain why not.

(d) (5 pts) If all else remains the same, but an external force of $f(t) = 30\cos 2t$ is applied to Slink, the resulting Laplace transform of the initial value problem is

$$X(s) = \frac{3}{s^2+4} - \frac{2}{s+1} + \frac{4}{s+4}$$

Find the amplitude of the steady state motion of Slink's head in this case.

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{\cosh bt\} = \frac{s}{s^2-b^2} \qquad \mathscr{L}\{\sinh bt\} = \frac{b}{s^2-b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{t f'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\text{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\text{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$