1. (21 pts) Parts (a) and (b) are unrelated.

   (a) Find the average value f_{ave} of the function $f(x) = \cos x - \sin x$ on the interval $[0, \pi/4]$.

   Solution:

$$\begin{aligned} f_{ave} &= \frac{1}{\frac{\pi}{4} - 0} \int_0^{\pi/4} (\cos x - \sin x)\, dx \\ &= \frac{4}{\pi} (\sin x + \cos x) \Big|_0^{\pi/4} \\ &= \frac{4}{\pi} \left(\sin(\pi/4) + \cos(\pi/4) - \sin(0) - \cos(0)\right) \\ &= \frac{4}{\pi} \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1 \right) \\ &= \boxed{\frac{4}{\pi} \left(\sqrt{2} - 1\right)} \end{aligned}$$

   (b) Find all values of x, if any, at which the value of the following function $u(x)$ is exactly equal to the average value of $u(x)$ on the specified interval. Fully simplify your answer. State "none" if none exist.

$$u(x) = x^{1/3}, \quad 0 \le x \le 8$$

   Solution:

$$\begin{aligned} u_{ave} &= \frac{1}{8 - 0} \int_0^8 x^{1/3}\, dx \\ &= \frac{1}{8} \cdot \frac{3}{4} x^{4/3} \Big|_0^8 \\ &= \frac{3}{32} \left(8^{4/3} - 0^{4/3}\right) \\ &= \frac{3}{32} (16 - 0) = \frac{3}{2} \end{aligned}$$

$$u(x) = x^{1/3} = 3/2$$

$$x = (3/2)^3 = \boxed{27/8}$$

2. (22 pts) Evaluate the following integrals.

(a) $\int x(x+3)^{1/4}\,dx$ (Express your answer in terms of x.)

Solution: Execute u-substitution with $u = x+3$, so that $du = dx$ and $x = u-3$.

$$\begin{aligned}
\int x(x+3)^{1/4}\,dx &= \int (u-3)\,u^{1/4}\,du \\
&= \int \left(u^{5/4} - 3u^{1/4}\right)\,du \\
&= \frac{4}{9}\,u^{9/4} - 3\cdot\frac{4}{5}\,u^{5/4} + C \\
&= \boxed{\frac{4}{9}\,(x+3)^{9/4} - \frac{12}{5}\,(x+3)^{5/4} + C}
\end{aligned}$$

(b) $\int_0^{\pi/3} \sin x\cdot\cos^2 x\,dx$ (Fully simplify your answer.)

Solution: Execute u-substitution with $u = \cos x$, so that $du = -\sin x\,dx$.

$x = 0$ corresponds to $u = \cos(0) = 1$ and $x = \pi/3$ corresponds to $u = \cos(\pi/3) = 1/2$.

$$\begin{aligned}
\int_0^{\pi/3} \sin x\cdot\cos^2 x\,dx &= -\int_1^{1/2} u^2\,du \\
&= -\frac{u^3}{3}\Bigg|_1^{1/2} \\
&= -\frac{1}{3}\left(\frac{1}{8} - 1\right) \\
&= \frac{1}{3}\cdot\frac{7}{8} = \boxed{\frac{7}{24}}
\end{aligned}$$

3. (17 pts) Parts (a) and (b) are unrelated.

(a) Find the inverse function of $g(x) = (8 - x^3)^{1/2}$ for $0 \le x \le 2$. You may assume that $g(x)$ is one-to-one.

Solution:

$$\begin{aligned} y &= (8 - x^3)^{1/2} \\ y^2 &= 8 - x^3 \\ x^3 &= 8 - y^2 \\ x &= (8 - y^2)^{1/3} \\ y &= (8 - x^2)^{1/3} \quad \text{(reverse the roles of } x \text{ and } y\text{)} \\ g^{-1}(x) &= \boxed{(8 - x^2)^{1/3}} \end{aligned}$$

(b) Consider the function $h(x) = x^7 + 3x^5 + 5x^3 + 7x + 9$.

i. Explain why h is invertible, based on its derivative.

ii. For what value of x does $h(x) = 9$?

iii. Determine the value of $(h^{-1})'(9)$. (*Hint:* Do not attempt to identify the function $h^{-1}(x)$.)

Solution:

i. $h'(x) = 7x^6 + 15x^4 + 15x^2 + 7$, which is positive for any real number x. So, $h(x)$ is $\boxed{\text{monotone}}$ increasing, and is therefore invertible.

ii. By inspection, $h(0) = 0^7 + 3 \cdot 0^5 + 5 \cdot 0^3 + 7 \cdot 0 + 9 = 9$. So, $h(x) = 9$ for $\boxed{x = 0}$

iii. From part (ii), the point $(0, 9)$ lies on the curve $y = h(x)$, so that the point $(9, 0)$ lies on the curve $y = h^{-1}(x)$. Therefore,

$$(h^{-1})'(9) = \frac{1}{h'(0)}$$

Using the expression for $h'(x)$ from part (i), we have $h'(0) = 7 \cdot 0^6 + 15 \cdot 0^4 + 15 \cdot 0^2 + 7 = 7$, so that

$$(h^{-1})'(9) = \boxed{\frac{1}{7}}$$

4. (24 pts) Parts (a) and (b) are unrelated.

(a) Evaluate $\displaystyle\int_0^2 \frac{x^3+1}{x^4+4x+1}\,dx$.

Solution: Execute u-substitution.

$$\begin{aligned} u &= x^4+4x+1 \\ \frac{du}{dx} &= 4x^3+4 = 4(x^3+1) \\ du &= 4(x^3+1)\,dx \end{aligned}$$

The limits of integration are transformed as follows:

$$\begin{aligned} x=0 \quad &\Rightarrow \quad u = 0^4+4\cdot 0+1 = 1 \\ x=2 \quad &\Rightarrow \quad u = 2^4+4\cdot 2+1 = 25 \end{aligned}$$

The transformed integral can now be constructed and evaluated.

$$\begin{aligned} \int_0^2 \frac{x^3+1}{x^4+4x+1}\,dx &= \frac{1}{4}\int_1^{25}\frac{du}{u} \\ &= \frac{1}{4}\cdot \ln|u|\,\Bigg|_1^{25} \\ &= \frac{1}{4}(\ln(25)-\ln(1)) \\ &= \boxed{\frac{1}{4}\ln(25) = \frac{1}{2}\ln(5)} \end{aligned}$$

(b) Find the derivative $y'(x)$ for the function $y = (\cos x + 2)^{\cos x + 2}$. Do not fully simplify your answer, although it must be expressed solely as a function of x.

Hint: Apply logarithmic differentiation.

Solution:

$$y = (\cos x + 2)^{\cos x + 2}$$

$$\ln y = \ln\left[(\cos x + 2)^{\cos x + 2}\right]$$

$$\ln y = (\cos x + 2)\ln(\cos x + 2)$$

$$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[(\cos x + 2)\ln(\cos x + 2)\right]$$

$$\frac{y'}{y} = (\cos x + 2)\cdot\frac{-\sin x}{\cos x + 2} + \ln(\cos x + 2)\cdot(-\sin x)$$

$$\frac{y'}{y} = -\sin x + \ln(\cos x + 2)\cdot(-\sin x)$$

$$\frac{y'}{y} = -\sin x[1 + \ln(\cos x + 2)]$$

$$y' = y(-\sin x)[1 + \ln(\cos x + 2)]$$

$$y'(x) = \boxed{\left[(\cos x + 2)^{\cos x + 2}\right](-\sin x)[1 + \ln(\cos x + 2)]}$$

5. (16 pts) Parts (a) and (b) are unrelated.

(a) Consider the function $w(x) = x^{-3} \cdot 3^{-x}$. Identify all values of x, if any, for which $w'(x) = 0$. If there are no such values, state "none" and explain why.

Solution:

$$\begin{aligned} w'(x) &= x^{-3} \cdot \frac{d}{dx}\left[3^{-x}\right] + 3^{-x} \cdot \frac{d}{dx}\left[x^{-3}\right] \\ &= x^{-3} \cdot \left[(3^{-x})(\ln 3)(-1)\right] + 3^{-x} \cdot (-3x^{-4}) \\ &= -3^{-x}x^{-4}(x \ln 3 + 3) \end{aligned}$$

$w'(x) = 0$ for x such that $x \ln 3 + 3 = 0$.

$$\begin{aligned} x \ln 3 + 3 &= 0 \\ x &= \boxed{-\frac{3}{\ln 3}} \end{aligned}$$

Alternate Solution: Logartithmic differentiation can be used to find $w'(x)$ as follows:

$$w = x^{-3} \cdot 3^{-x}$$

$$\begin{aligned} \ln w &= \ln\left(x^{-3} \cdot 3^{-x}\right) \\ &= \ln\left(x^{-3}\right) + \ln\left(3^{-x}\right) \\ &= -3 \ln x - x \ln(3) \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}[\ln w] &= \frac{d}{dx}[-3 \ln x] - \frac{d}{dx}[x \ln(3)] \\ \frac{w'}{w} &= -\frac{3}{x} - \ln(3) \end{aligned}$$

$$\begin{aligned} w' &= w \cdot \left(-\frac{1}{x}\right) \cdot (3 + x \ln(3)) \\ w'(x) &= \left(x^{-3} \cdot 3^{-x}\right) \cdot \left(-\frac{1}{x}\right) \cdot (3 + x \ln(3)) \\ &= -x^{-4}\, 3^{-x}(3 + x \ln 3) \end{aligned}$$

(b) Solve the following equation for t: $\quad 2 = \dfrac{7}{3 + 5e^{-2t}}$.

Solution:

$$\begin{aligned}
6 + 10e^{-2t} &= 7 \\
10e^{-2t} &= 1 \\
e^{-2t} &= 0.1 \\
-2t &= \ln(0.1) \\
t &= \boxed{-\frac{1}{2}\ln(0.1) = \frac{\ln(10)}{2}}
\end{aligned}$$