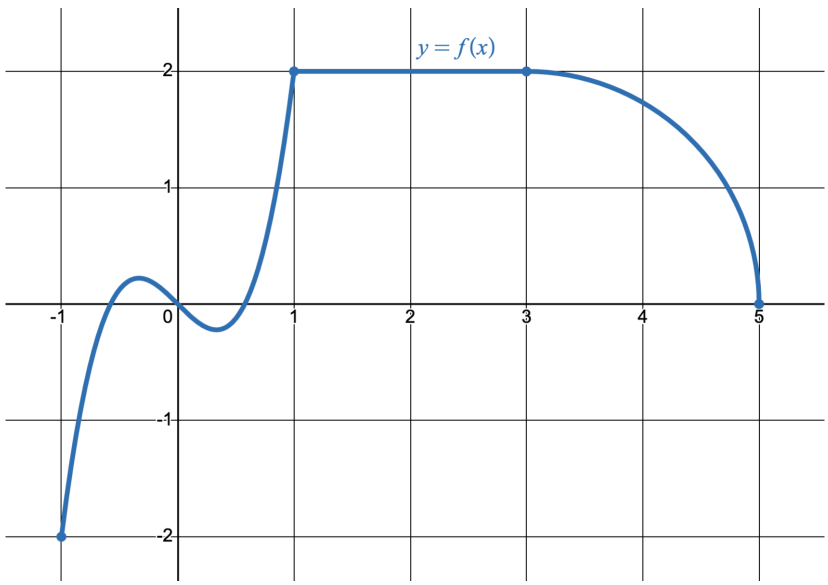
1. (32 points) The following parts are not related.

   (a) Evaluate: $\displaystyle\int \left[\frac{\sqrt{x}}{x^3} + 2\sec(x)\tan(x)\right]\, dx$

   (b) Evaluate: $\displaystyle\frac{d}{dx}\int_3^{x^3} \frac{\tan\left(t^2\right)}{t^4}\, dt$

   (c) Evaluate: $\displaystyle\int_{-2}^{4} h(x)\, dx$ if we know $\displaystyle\int_{-2}^{4} \left(g(x)+h(x)\right)\, dx = 4$ and $\displaystyle\int_{4}^{-2} 3g(x)\, dx = 8$

   (d) Evaluate: $\displaystyle\int_{-1}^{5} f(x)\, dx$ for the function $y = f(x)$ graphed below. Note that the graph consists of an odd function between $x=-1$ and $x=1$, a line segment between $x=1$ and $x=3$, and a quarter circle between $x=3$ and $x=5$. You may use geometry or a combination of geometry and integration to compute your answer.



Solution:

(a)

$$\int \left[\frac{\sqrt{x}}{x^3} + 2\sec(x)\tan(x)\right]\, dx = \int x^{-5/2} + 2\sec(x)\tan(x) dx$$
$$= \frac{x^{-3/2}}{-3/2} + 2\sec(x) + C$$
$$= \boxed{-\frac{2}{3}x^{-3/2} + 2\sec(x) + C}$$

(b) $\displaystyle\frac{d}{dx}\int_3^{x^3} \frac{\tan\left(t^2\right)}{t^4}\, dt = \boxed{\frac{\tan(x^6)}{x^{12}}\cdot(3x^2)}$

(c) We are given that $\displaystyle\int_{-2}^{4} (g(x)+h(x))\ dx = 4$ and $\displaystyle\int_{4}^{-2} 3g(x)\ dx = 8$. Taking the second integral, we see that:

$$\int_{4}^{-2} 3g(x)\ dx = -\int_{-2}^{4} 3g(x)\ dx = -3\int_{-2}^{4} g(x)\ dx$$

Therefore, we have that

$$-3\int_{-2}^{4} g(x)\ dx = 8 \implies \int_{-2}^{4} g(x)\ dx = -\frac{8}{3}$$

Next, note that

$$\int_{-2}^{4} (g(x)+h(x))\ dx = \int_{-2}^{4} g(x)\ dx + \int_{-2}^{4} h(x)\ dx$$

Therefore, we have that $\int_{-2}^{4} g(x)\ dx + \int_{-2}^{4} h(x)\ dx = 4$

Since we know from above that the integral of $g(x)$ on this interval is $-8/3$, we can solve for the integral of $h(x)$ by doing the following:

$$-\frac{8}{3} + \int_{-2}^{4} h(x)dx = 4 \implies \boxed{\int_{-2}^{4} h(x)dx = \frac{20}{3}}$$

(d) Since the function varies as an odd function on the interval $[-1,1]$, a constant function on the interval $[1,3]$, and a quarter circle on the interval $[3,5]$, we will split up our given interval into three separate integrals on those intervals and solve each one with geometry.

$$\int_{-1}^{5} f(x)\ dx = \int_{-1}^{1} f(x)dx + \int_{1}^{3} f(x)dx + \int_{3}^{5} f(x)dx$$

Since $f(x$ is odd on the interval $[-1,1]$, we see that $\displaystyle\int_{-1}^{1} f(x)dx = 0$.

Since $f(x) = 2$ on the interval $[1,3]$, we are finding the area of a square over this interval. Therefore, $\displaystyle\int_{1}^{3} f(x)dx = 4$.

Lastly, note that we have a quarter circle over the last interval, $[3,5]$. This quarter circle has a radius of 2. Therefore, $\displaystyle\int_{3}^{5} f(x)dx = \frac{\pi \cdot 2^2}{4} = \pi$.

Putting this all together we have that,

$$\int_{-1}^{5} f(x)\ dx = 0 + 4 + \pi = \boxed{4+\pi}$$

2. (20 points) Consider a function $f(x) = x^2 - 4x - 5$, which is referenced in each part of this problem.

   (a) Verify that $f(x)$ satisfies the hypotheses of Rolle's Theorem on $[-1, 5]$. Then, determine the value(s) of c at which $f'(c) = 0$ in $(-1, 5)$.

   (b) Assuming that $f(x)$ satisfies the hypotheses of the Mean Value Theorem on $[1, 4]$, determine all value(s) of c in $(1, 4)$ that satisfy the conclusion of the Mean Value Theorem.

Solution: Since $f(x) = x^2 - 4x - 5$ is a polynomial, it is continuous and differentiable for all real x.

(a) To apply Rolle's Theorem on $[-1, 5]$, we need

$$f(-1) = f(5).$$

First compute

$$f(-1) = (-1)^2 - 4(-1) - 5 = 1 + 4 - 5 = 0.$$
$$f(5) = 5^2 - 4(5) - 5 = 25 - 20 - 5 = 0.$$

So $f(-1) = f(5)$, and the hypotheses of Rolle's Theorem are satisfied.
Rolle's Theorem guarantees at least one $c \in (-1, 5)$ such that $f'(c) = 0$. Compute the derivative:

$$f'(x) = 2x - 4.$$

So $f'(x) = 0$ when

$$2x - 4 = 0 \qquad \Rightarrow \qquad x = 2.$$

Since $2 \in (-1, 5)$, the value is

$$\boxed{c = 2}.$$

(b) Since $f(x)$ is a polynomial, it is continuous on $[1, 4]$ and differentiable on $(1, 4)$, so the hypotheses of the Mean Value Theorem are satisfied.
The average rate of change on $[1, 4]$ is

$$f_{avg} = \frac{f(4) - f(1)}{4 - 1}.$$

Compute:

$$f(4) = 4^2 - 4(4) - 5 = 16 - 16 - 5 = -5,$$
$$f(1) = 1^2 - 4(1) - 5 = 1 - 4 - 5 = -8.$$

Therefore,

$$\frac{f(4) - f(1)}{4 - 1} = \frac{-5 - (-8)}{3} = \frac{3}{3} = 1.$$

We now solve

$$f'(c) = 1.$$

Since $f'(x) = 2x - 4$, we get

$$2c - 4 = 1,$$

so

$$2c = 5 \qquad \Rightarrow \qquad c = \frac{5}{2}.$$

Since $\frac{5}{2} \in (1, 4)$, we have

$$\boxed{c = \frac{5}{2}}.$$

3. (22 points) Consider the function

$$f(x) = \frac{1}{x(x-3)^2} \text{ with } f'(x) = \frac{3(1-x)}{x^2(x-3)^3} \text{ and } f''(x) = \frac{6(2x^2-4x+3)}{x^3(x-3)^4}$$

(a) Does the function have vertical asymptotes? Justify your answer using appropriate limits.

(b) Does the function have horizontal asymptotes? Justify your answer using appropriate limits.

(c) Find the x coordinates of the local maxima and minima, if any.

(d) Find the interval(s) where $f(x)$ is concave up, and the interval(s) where it is concave down. Hint: The numerator of $f''(x)$ cannot be factored in the real numbers and is always positive.

(e) Sketch a graph of $f(x)$. Clearly label intercepts, asymptotes, and local extrema.

Solution:

(a) (4 pts) Does the function have vertical asymptotes? Justify your answer using appropriate limits.
Soln:

$$\lim_{x\to 0-} \frac{1}{x(x-3)^2} = -\infty \text{ and } \lim_{x\to 0+} \frac{1}{x(x-3)^2} = \infty$$
$$\lim_{x\to 3-} \frac{1}{x(x-3)^2} = \infty \text{ and } \lim_{x\to 3+} \frac{1}{x(x-3)^2} = \infty$$

Hence $\boxed{x = 0}$ and $\boxed{x = 3}$ are vertical asymptotes.

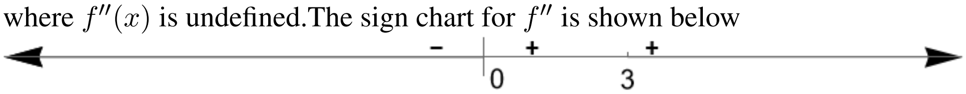
(b) (4 pts) Does the function have horizontal asymptotes? Justify your answer using appropriate limits.
Soln:

$$\lim_{x\to\infty} \frac{1}{x(x-3)^2} = 0$$

Hence $\boxed{y = 0}$ is a horizontal asymptote.

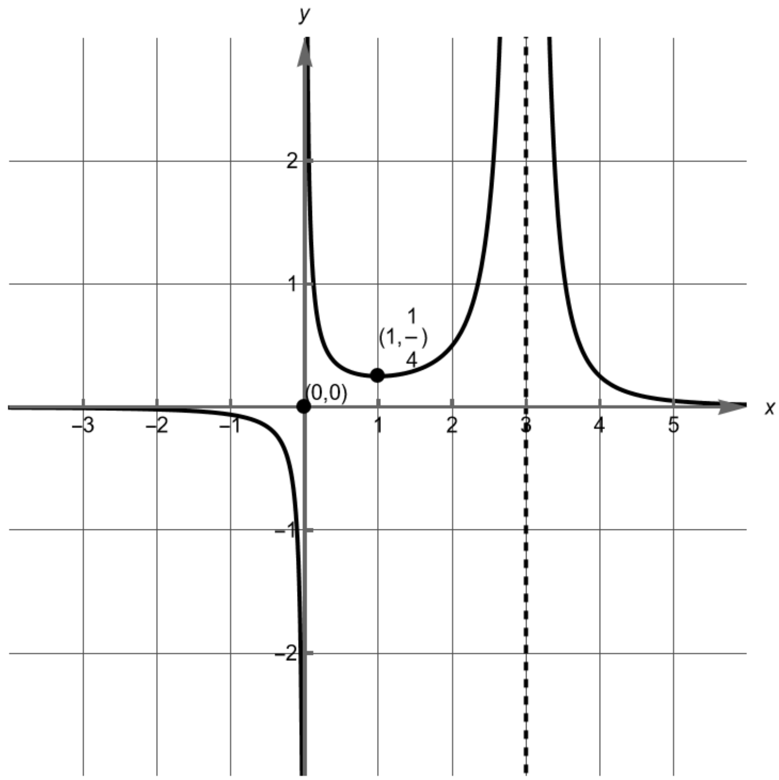
(c) (4 pts) Find the x coordinates of the local maxima and minima, if any.
Soln:
$f'(x) = 0$ at $x = 1$. Also, $f''(1) = \frac{3}{8} > 0$.
Hence, by the 2nd derivative test, there is a $\boxed{\text{minimum at } x = 1}$

(d) (4 pts) Find the interval(s) where $f(x)$ is concave up, and the interval(s) where it is concave down. Hint: The numerator of $f''(x)$ cannot be factored in the real numbers and is always positive.
Soln:
Since the numerator of $f''(x)$ cannot be factored in the real numbers and is always positive, f'' doesn't have any zero. However, we still have to check the concavity around the vertical asymptotes $x = 0$ and $x = 3$, where $f''(x)$ is undefined.The sign chart for f'' is shown below



Hence f is $\boxed{\text{concave down in}(\infty, 0)}$ and $\boxed{\text{concave up in } (0,3) \cup (3,\infty)}$

(e) (6 pts) Sketch a graph of $f(x)$ Clearly label intercepts, asymptotes, and local extrema.


y
x
2
1
–1
–2
–3
–2
–1
1
2
3
4
5
(0,0)
(1, 1/4)

4. (14 points) All questions below will consider the function $f(x) = \sin x + x - 1$.

   (a) Given an initial guess of $x_1 = \frac{\pi}{2}$, compute the first two iterations of Newton's method to find the root of $f(x)$. In other words, find x_3.

   (b) What is the Riemann sum with n rectangles that would be used to approximate the area between $f(x)$ and the x-axis on the interval from $[0, \pi]$. Leave your answer in terms of a summation (sigma notation). Make sure to explicitly define Δx and $f(x_i)$ for this particular function.

Solution:

(a) The formula for Newton's Method is: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. Our function in particular along with it's derivative is:

$$f(x) = \sin x + x - 1$$
$$f'(x) = \cos x + 1$$

Therefore, the Newton's method formula we shall use is:

$$x_{n+1} = x_n - \frac{\sin x_n + x_n - 1}{\cos x_n + 1}$$

.

Since we are given that $x_1 = \frac{\pi}{2}$, then we have that:

$$\begin{aligned} x_2 &= x_1 - \frac{\sin x_1 + x_1 - 1}{\cos x_1 + 1} \\ &= \frac{\pi}{2} - \frac{\sin \frac{\pi}{2} + \frac{\pi}{2} - 1}{\cos \frac{\pi}{2} + 1} \\ &= \frac{\pi}{2} - \frac{1 + \frac{\pi}{2} - 1}{0 + 1} \\ &= \frac{\pi}{2} - \frac{\pi}{2} \\ &= 0 \end{aligned}$$

Repeating the procedure to compute x_3, we have:

$$\begin{aligned} x_3 &= x_2 - \frac{\sin x_2 + x_2 - 1}{\cos x_2 + 1} \\ &= 0 - \frac{\sin 0 + 0 - 1}{\cos 0 + 1} \\ &= 0 - \frac{-1}{2} \\ &= \frac{1}{2} \end{aligned}$$

Thus, we have found that $\boxed{x_2 = 0}$ and $\boxed{x_3 = 1/2}$.

(b) The general form of the Riemann sum is: $\sum_{i=1}^{n} f(x_i)\Delta x$.

For our case, $\Delta x = \frac{\pi - 0}{n}$.

Since we define $x_i = a + i\Delta x$, then $x_i = 0 + i \times \frac{\pi}{n}$.

Therefore, $f(x_i) = \sin \frac{\pi}{n} \cdot i + \left(\frac{\pi}{n} \cdot i\right) - 1$.

Putting this altogether, we have:

$$\boxed{R_n = \sum_{i=1}^{n} \left[\left(\sin\left(\frac{\pi}{n} \cdot i\right) + \left(\frac{\pi}{n} \cdot i\right) - 1\right) \cdot \frac{\pi}{n}\right]}$$

5. (12 points) Suppose an airline policy states that all baggage must be box-shaped with a sum of the length, width, and height not exceeding 64 inches. What are the **dimensions** and **volume** of a square-based box (i.e. the base of the box is square) with the greatest volume under these conditions? Make sure to label the dimensions of the box and to report the maximum volume. You must use calculus and the optimization methods covered in our class to solve this problem. Show all work and box your answers.

Solution:

We seek to maximize the volume of a box, $V = l \times w \times h$, given the constraint that $l + w + h = 64$. Additionally, we know that since the base of the box is square, we have that $l = w$.

So, we can express the volume as $V = w^2 h$ and the constraint it $2w + h = 64$.

Using the constraint relationship, we can solve for h, to get:

$$h = 64 - 2w$$

Then, we will plug that expression into the volume equation to achieve a volume function in terms of w:

$$V(w) = w^2(64 - 2w) = 64w^2 - 2w^3$$

Next, we compute the first derivative of $V(w)$ and find any critical points:

$$V'(w) = 128w - 6w^2 = 0$$

Solving this for w, we have

$$128w - 6w^2 = 2w(64 - 3w) = 0$$

$\implies w = 0, w = \frac{64}{3}$

Since $w > 0$, as w represents a width, our only critical point is $w = \frac{64}{3}$

To prove that this critical point is the local of a local maximum, we will use the first derivative test.

Since $0 < 1 < \frac{64}{3}$, we compute $V'(1) = 128 - 6 = 122 > 0$.

Since $32 > 30 > \frac{64}{3}$, we compute $V'(30) = 3 \times 1280 - 5400 < 0$

Alternatively, by the second derivative test, $V''(w) = 128 - 12w$. Since $V''(\frac{64}{3}) = 128 - 4 \times 64 = -128 < 0$, we know that the volume function is concave down at $w = \frac{64}{3}$.

By either test, we have shown that $w = \frac{64}{3}$ is a maximum.

The dimensions of the box are then: $\boxed{w = \frac{64}{3}, l = \frac{64}{3}, h = 64 - \frac{2}{3} \cdot 64 = \frac{64}{3}}$.

The volume of the box is: $\boxed{V = \frac{64^3}{27}}$.