Write your name and your professor's name or your section number below. You are *not* allowed to use textbooks, notes, or a calculator. To receive full credit on a problem you must show **sufficient justification for your conclusion** unless explicitly stated otherwise.

This exam has 5 problems and is worth 100 points.

Name:

Instructor and Section:

1. (24 pts, 6 each) If the statement is **always true**, write "TRUE"; if it is possible for the statement to be false then mark "FALSE." You must give a **justification** for your answer. That is, if the answer is true, provide a brief proof. If the answer is false, provide a counterexample.

   (a) If A is square and $\det A = 0$ then the equation $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

   (b) Multiplying a column of a matrix by a real number also multiplies the determinant by the same real number.

   (c) The set of 2×2 matrices with a 0 in the $2, 1$ entry is a subspace of $\mathbb{R}^{2\times 2}$ with dimension 3.

   (d) Let A be a 4×4 matrix. If $A^2 = A$, then either $A = I$ or $A = 0$ (the zero matrix).

Solution

(a) **False.** A counterexample would be

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The determinant of the matrix is 0 but the system has no solutions.

(b) **True.** Since multiplying a row of a matrix by c multiplies the determinant by c and the determinant of the transpose equals the determinant of the original matrix.

(c) **True.** The set contains $O_{2\times 2}$ so is not empty, the addition of two members is also a member and multiplying a member by a real number produces another member, so it is closed under both operations. Finally, the standard basis for this subspace is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

which has 3 basis vectors, so the dimension of the subspace is 3.

(d) **False.** Here's a counterexample where $A^2 = A$, but $A \neq I$ and $A \neq 0$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. (20 pts) Let

$$A = \begin{pmatrix} -1 & 2 & 8 \\ 2 & 3 & 5 \\ 0 & -1 & -3 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} -5 \\ 3 \\ -1 \end{pmatrix}$$

(a) (8 pts) Find the Row Echelon Form of A

(b) (2 pts) What is the rank of A?

(c) (6 pts) Does $A\mathbf{x} = \mathbf{b}$ have a solution? If so, is there more than one solution?

(d) (4 pts) Are there any vectors $\mathbf{c}$ such that $A\mathbf{x} = \mathbf{c}$ has infinitely many solutions?

Solution

(a)

$$\begin{pmatrix} -1 & 2 & 8 \\ 2 & 3 & 5 \\ 0 & -1 & -3 \end{pmatrix} \xrightarrow[R_2 \to R_2 + 2R_1]{} \begin{pmatrix} -1 & 2 & 8 \\ 0 & 7 & 21 \\ 0 & -1 & -3 \end{pmatrix}$$

$$\xrightarrow[R_3 \to R_3 + (1/7)R_2]{} \begin{pmatrix} -1 & 2 & 8 \\ 0 & 7 & 21 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) As there are 2 pivot columns in the REF, the rank of A is 2

(c) We need to form the augmented matrix and then perform the same row operations as in part (a):

$$\left(\begin{array}{ccc|c} -1 & 2 & 8 & -5 \\ 2 & 3 & 5 & 3 \\ 0 & -1 & -3 & -1 \end{array}\right) \xrightarrow[R_2 \to R_2 + 2R_1]{} \left(\begin{array}{ccc|c} -1 & 2 & 8 & -5 \\ 0 & 7 & 21 & -7 \\ 0 & -1 & -3 & -1 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 + (1/7)R_2]{} \left(\begin{array}{ccc|c} -1 & 2 & 8 & -5 \\ 0 & 7 & 21 & -7 \\ 0 & 0 & 0 & -2 \end{array}\right)$$

Our system is inconsistent due to the -2 in the bottom row. There is therefor no solution.

(d) Yes, since there is a free variable in the system, any vector $\mathbf{c}$ that has one solution has infinitely many solutions.

3. (15 pts) Consider the set of vectors

$$V = \left\{ \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 11 \end{pmatrix} \right\}.$$

(a) (7 points) Are these vectors linearly dependent or linearly independent? Justify your answer.

(b) (4 points) Find a basis for the span of V and state its dimension.

(c) (4 points) Is the vector $\begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$ in the span of V?

Solution:

(a) For matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -1 & 3 \\ 4 & 1 & 11 \end{pmatrix},$$

we construct row echelon form.

$$\begin{pmatrix} 2 & -1 & 1 \\ 3 & -1 & 3 \\ 4 & 1 & 11 \end{pmatrix} \xrightarrow[R_2 \to R_2 - \frac{3}{2}R_1]{} \begin{pmatrix} 2 & -1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 4 & 1 & 11 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{} \begin{pmatrix} 2 & -1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 3 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 3 & 9 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 6R_2]{} \begin{pmatrix} 2 & -1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

The vectors are linearly dependent since the matrix A is singular.

(b) The dimension of the span is 2 and is equal to the rank of A. The first two vectors are linearly independent and form a basis.

(c) Checking if the given vector can be written as a linear combination of the two basis vectors. We seek c_1 and c_2 such that

$$c_1 \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & -1 & 3 \\ 3 & -1 & -3 \\ 4 & 1 & 1 \end{array}\right) \xrightarrow[R_2 \to R_2 - \frac{3}{2}R_1]{} \left(\begin{array}{cc|c} 2 & -1 & 3 \\ 0 & -\frac{1}{2} & -\frac{15}{2} \\ 4 & 1 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{} \left(\begin{array}{cc|c} 2 & -1 & 3 \\ 0 & -\frac{1}{2} & -\frac{15}{2} \\ 0 & 3 & -5 \end{array}\right)$$

so that

$$-\frac{1}{2}c_2 = -\frac{15}{2} \quad \text{and} \quad 3c_2 = -5,$$

which implies that there is no solution and, therefore, the vector is not in the span of V.

4. (20 pts) The following questions are unrelated.

   (a) (5 pts) For a regular matrix A that admits an LU decomposition, what is the determinant of UL^T?

   (b) (10 pts) Consider the matrix

$$B = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

   Find the LDL^T decomposition of B.

   (c) (5 pts) For the following system

$$P \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 5 \\ 2 \\ 4 \end{pmatrix},$$

   what is a matrix P that makes this equation true?

Solution:

(a) The determinant of UL^T equals the determinant of the original matrix A. This follows from the fact that

$$|UL^T| = |U| \cdot |L^T| = |L^T| \cdot |U| = |L| \cdot |U| = |LU| = |A|.$$

(b) The decomposition of B is:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(c) For this problem:

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

5. (21 pts)

Consider matrix

$$A = \begin{pmatrix} 0 & r & -s \\ -r & 0 & t \\ s & -t & 0 \end{pmatrix}$$

and a linear system

$$A\mathbf{x} = \mathbf{b},$$

where

$$\mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

(a) (7 pts) Is the matrix A invertible? Justify your answer.

(b) (7 pts) What is the rank of A if $r = s = t = 1$? Justify your answer.

(c) (7 pts) Find the general solution of $A\mathbf{x} = \mathbf{b}$ assuming $r = s = t = 1$.

Solution:

(a) Computing the determinant directly, we observe $\det(A) = 0$ and, therefore, the matrix is not invertible.

(b) The first two columns (or the two rows) of the matrix A

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

are linearly independent so that the rank is equal to 2.

(c) Computing the row echelon form, we have

$$\left(\begin{array}{ccc|c} 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{array}\right) \xrightarrow[R_1 \to R_2, R_2 \to R_1]{} \left(\begin{array}{ccc|c} -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 + R_1]{}$$

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 + R_2]{} \left(\begin{array}{ccc|c} -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

A solution of the homogeneous system $Ax = 0$ is $x = (1, 1, 1)^T$. Combined with a particular solution, e.g. $(1, 1, 2)^T$, the general solution can be written as

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$