APPM 3570/STAT 3100 Exam 1 Spring 2026

NAME: ____

SECTION: 001 [*or*] 002

Instructions:

1. A calculator and a 3×5 note card are permitted.
2. Your text and other books, cell phones, and other electronic devices are not permitted during the exam.
3. Justify your answers, show all work.
4. When you have completed the exam, go to the uploading area in the room and scan your exam and upload it to Gradescope.
5. Don't forget to scan any back pages you used for extra space!
6. Verify that everything has been uploaded correctly and the pages have been associated to the correct problems.
7. Turn in your hardcopy exam.

On my honor as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work.

Signature: ______________________________ Date: ________________

Duration: 90 minutes

Problem 1. (16 points) The following four questions are unrelated.

(a) How many arrangements can be formed from the letters in GOOGLE?

(b) Nine people (including Alice and Bob) are available to serve on a committee of five people. In how many ways can this committee be formed if Alice and Bob do not want to be on the committee together?

(c) There are seven parking spaces in a row. In how many ways can four distinct cars park in the seven spaces? (Assume that every car parks head-in.)

(d) Determine the value of $\binom{13}{0} + \binom{13}{1} + \binom{13}{2} + \cdots + \binom{13}{12} + \binom{13}{13}$.

Solution:

(a) (4 points.) $\binom{6}{2,2} = \frac{6!}{2!\,2!} = 180$ arrangements.

(b) (4 points) $\binom{2}{1}\binom{7}{4} + \binom{2}{0}\binom{7}{5} = 91$ ways.

(c) (4 points.) $7!/3! = 840$ ways.

(d) (4 points.) By the binomial theorem, the value equals $(1+1)^{13} = 2^{13} = 8192$.

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Problem 2. (24 points.) The following three questions are unrelated.

(a) Let A, B, C be independent events. Is $A \cap B$ independent of C? Is $A \cup B$ independent of C? Justify your answers with a mathematical argument or counterexample.

(b) A standard deck of poker cards has 52 cards, with 4 suits (hearts, diamonds, spades and clubs) and 13 denominations (Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, and King) in each suit. If three cards are drawn one by one (without replacement) from a fully shuffled standard deck of 52 cards, what is the probability that an Ace (A) appears before a King (K) does?

(c) A special die comes up odd numbers with probability 2/3 and all the possible odd numbers (i.e., 1,3, and 5) share equal probability. The die comes up even numbers with probability 1/3 and all the possible even numbers (i.e., 2,4, and 6) share equal probability. If we keep rolling this die indefinitely, what is the probability that we see a 6 before we see any odd number?

Solution:

(a) (8 points.) Since A, B, C are independent,

$$P(AB|C) = \frac{P(ABC)}{P(C)} = \frac{P(A)P(B)P(C)}{P(C)} = P(A)P(B) = P(AB). \tag{1}$$

Hence, AB and C are independent.

Now, using the conditional version of the inclusion-exclusion formula, A being independent of C, B being independent of C, and (1), we get

$$\begin{aligned} P(A \cup B|C) &= P(A|C) + P(B|C) - P(AB|C) \\ &= P(A) + P(B) - P(AB) = P(A \cup B), \end{aligned}$$

where the last equality follows from the standard (unconditional) inclusion-exclusion formula. Hence, $A \cup B$ and C are independent.

(b) (8 points.) Let E be the event that A appears before K. Note that $E = E_1 \cup E_2 \cup E_3$, where E_1 is the event that the first card is an A, E_2 is the event that the first card is neither an A nor K while the second card is an A, and E_3 is the event that the first two cards are neither an A nor K while the third card is an A. As E_1, E_2, E_3 are disjoint by definition,

$$\begin{aligned} \mathrm{P}(E) = \mathrm{P}(E_1) + \mathrm{P}(E_2) + \mathrm{P}(E_3) &= \frac{4 \cdot 51 \cdot 50 + 44 \cdot 4 \cdot 50 + 44 \cdot 43 \cdot 4}{52 \cdot 51 \cdot 50} \\ &= \frac{1107}{5525} = 0.2004. \end{aligned}$$

(c) (8 points.) First, in any given roll of the die, each of the three even numbers has equal probability $(1/3)/3 = 1/9$. Let E be the event that we see a 6 before we see any odd number. Note that $E = E_1 \cup E_2 \cup E_3 \cup \cdots$, where E_1 is the event that the first roll yields 6, E_2 is the event that the first roll yields 2 or 4 while the second roll yields 6, E_3 is the event that the first two rolls yields 2 or 4 while the third roll yields 6. In general, E_n is the event that the first $(n-1)$ rolls yield 2 or 4 while the n^{th} roll yields 6. As $E_1, E_2, E_3, \ldots$ are disjoint by definition,

$$\mathrm{P}(E) = \sum_{n=1}^{\infty} \mathrm{P}(E_n) = \sum_{n=1}^{\infty} \left(\frac{2}{9}\right)^{n-1} \frac{1}{9} = \frac{1}{9} \sum_{n=1}^{\infty} \left(\frac{2}{9}\right)^{n-1} = \frac{1}{9} \frac{1}{1 - \frac{2}{9}} = \frac{1}{9} \cdot \frac{9}{7} = \frac{1}{7}.$$

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Problem 3. (30 points) An artificial intelligence (AI) test is developed for lung images and it gives either a positive or negative result for lung cancer. Given that a person has lung cancer, the probability of a positive AI test result is 0.8. Given that a person does not have lung cancer, the probability of a negative AI test result is 0.9. Suppose that each person in the population has lung cancer with probability 0.05.

(a) If three people are selected independently from the population, what is the probability that at least one of them has lung cancer?

(b) If a person is randomly selected to take the AI test, what is the probability of a positive test result?

(c) Given that a person gets a positive AI test result, what is the probability that this person really has lung cancer?

(d) A standard test for lung cancer involves visual examination of lung images by experienced medical doctors, and it also gives either a positive or negative result. Given that a person has lung cancer, a positive standard test result comes up with probability 0.75, independently of the AI test result. Given that a person does not have lung cancer, a negative standard test result comes up with probability 0.95, independently of the AI test result. If a person has positive results on both the AI and standard tests, what is the probability that this person really has lung cancer?

(e) Given your answers in parts (c) and (d), explain whether it is important to have a second test for lung cancer.

Solution:

(a) (6 points) P(at least one has lung cancer) $= 1 -$ P(no one has lung cancer) $= 1 - (0.95)^3 =$ 0.1426.

(b) (6 points)

$$\begin{aligned}\mathrm{P(positive)} &= \mathrm{P(positive \mid has\ cancer)P(has\ cancer)} + \mathrm{P(positive \mid no\ cancer)P(no\ cancer)} \\ &= (0.8)(0.05) + (0.1)(0.95) = 0.135.\end{aligned}$$

(c) (6 points) By Bayes' formula, P(has cancer | positive) equals

$$\frac{\mathrm{P(positive \mid has\ cancer)P(has\ cancer)}}{\mathrm{P(positive \mid has\ cancer)P(has\ cancer)} + \mathrm{P(positive \mid no\ cancer)P(no\ cancer)}}$$
$$= \frac{(0.8)(0.05)}{(0.8)(0.05) + (0.1)(0.95)} = 0.2963.$$

(d) (8 points) Let P_1 (resp. P_2) denote the event that the person gets a positive result in the AI (resp. standard) test. Let C denote the event that the person has lung cancer. Then,

$$\begin{aligned}\mathrm{P}(C \mid P_1P_2) &= \frac{\mathrm{P}(P_1P_2 \mid C)\mathrm{P}(C)}{\mathrm{P}(P_1P_2 \mid C)\mathrm{P}(C) + \mathrm{P}(P_1P_2 \mid C^c)\mathrm{P}(C^c)} \\ &= \frac{(0.8)(0.75)(0.05)}{(0.8)(0.75)(0.05) + (0.1)(0.05)(0.95)} = 0.8633.\end{aligned}$$

(e) (4 points) It is important to have a second test in the present context. Given that a person has lung cancer, the AI test can correctly detect this with probability as low as 0.2963; when we rely on both the AI and standard tests, the probability of correct detection rises to 0.8633.

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Problem 4. (30 points.) An urn contains four red balls and three white balls. All the seven balls feel the same and cannot be distinguished when we don't look at them. Suppose that we choose four balls from the urn simultaneously (without looking) at random.

(a) What is the probability that there are exactly two red balls within the chosen four?

(b) Within the chosen four balls, let G be the number of red balls times the number of white balls. Determine the probability mass function (pmf) of G.

(c) Determine the cumulative distribution function (cdf) of G.

(d) Find $\mathrm{P}(0 \le G \le 3)$.

Solution:

(a) (5 points.) $\frac{\binom{4}{2}\binom{3}{2}}{\binom{7}{4}} = \frac{18}{35}$.

(b) (12 points.) Let R denote the number of red balls within the chosen four. Observe that $G = R \cdot (4 - R) = 4R - R^2$. Since $R \in \{1, 2, 3, 4\}$, we find that $G \in \{0, 3, 4\}$ with

$$P(G = 0) = P(R = 4) = \frac{1}{\binom{7}{4}} = \frac{1}{35},$$

$$P(G = 3) = P(R = 1) + P(R = 3) = \frac{\binom{4}{1}\binom{3}{3}}{\binom{7}{4}} + \frac{\binom{4}{3}\binom{3}{1}}{\binom{7}{4}} = \frac{4 + 12}{35} = \frac{16}{35},$$

$$P(G = 4) = P(R = 2) = \frac{18}{35} \quad \text{(by part (a))}.$$

Thus, the pmf of G is

$$P(G = a) = \begin{cases} 1/35, & \text{if } a = 0; \\ 16/35, & \text{if } a = 3; \\ 18/35, & \text{if } a = 4; \\ 0, & \text{otherwise.} \end{cases}$$

(c) (8 points.) From part (b), we find that the cdf of G is

$$F_G(a) = \mathrm{P}(G \le a) = \begin{cases} 0, & \text{for } a < 0; \\ 1/35, & \text{for } 0 \le a < 3; \\ 17/35, & \text{for } 3 \le g < 4; \\ 1, & \text{for } g \ge 4. \end{cases}$$

(d) (5 points.)

$$\mathrm{P}(0 \le G \le 3) = \mathrm{P}(G = 0) + \mathrm{P}(G = 1) + \mathrm{P}(G = 2) + \mathrm{P}(G = 3) = 1/35 + 0 + 0 + 16/35 = 17/35.$$

Alternatively,

$$\mathrm{P}(0 \le G \le 3) = \mathrm{P}(G \le 3) - \mathrm{P}(G < 0) = F_G(3) - F_G(0-) = 17/35 - 0 = 17/35.$$

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