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1. An unusual region has its elevation given by $z = f(x, y) = y^3 + 3x^2y - 6x^2 - 6y^2$, where $x, y,$ and z are all measured in meters.

   (a) (14 points) Determine the (x, y)-coordinates of any peaks (local maximums), valleys (local minimums), or saddle points. (Be sure to determine which of the three occurs at each such point.)

   (b) (9 points) A small South American deer called a *pudu* stands at the point $(2, 1)$. From this point, what is the **unit vector** that points in the direction in which the pudu should head so its path will be the steepest uphill climb?

   (c) (9 points) The pudu starts to travel such that the (x, y)-coordinates of its location are given by

   $$\mathbf{r}(t) = \langle 2 - t, 1 - \frac{1}{2}t^2 \rangle$$

   after t minutes.

      i. Find the rate of change of the pudu's elevation with respect to *time* when it is at the point $(0, -1)$.
      ii. Find the rate of change of the pudu's elevation with respect to *distance* when it is at the point $(0, -1)$.
      iii. Is the pudu heading **uphill** or **downhill** at the point $(0, -1)$?

Solution:

(a) First, we must find any critical points. So, we find the first partial derivatives and determine where they are simultaneously zero:

$$f_x = 6xy - 12x = 6x(y - 2) = 0 \implies x = 0 \text{ or } y = 2$$

and

$$f_y = 3y^2 + 3x^2 - 12y = 0.$$

If $x = 0$, then

$$f_y = 3y^2 + 3(0)^2 - 12y = 3y(y - 4) = 0 \implies y = 0 \text{ or } y = 4.$$

If $y = 2$, then

$$f_y = 3(2)^2 + 3x^2 - 12(2) = 3(x^2 - 4) = 0 \implies x = \pm 2.$$

So, the critical points are $(0, 0)$, $(0, 4)$, and $(\pm 2, 2)$.

We will now apply the second derivative test to determine which of these the location of a local maximum, local minimum, or a saddle point.

The second derivative are

$$f_{xx} = 6y - 12 = f_{yy} \text{ and } f_{xy} = f_{yx} = 6x.$$

So we have

$$D = f_{xx}f_{yy} - f_{xy}^2 = (6y - 12)^2 - 36x^2.$$

Analyzing each points, we have

- $D(0, 0) > 0$ and $f_{xx}(0, 0) < 0$, so $(0, 0)$ is the location of a local maximum.

- $D(0,4) > 0$ and $f_{xx}(0,4) > 0$, so $(0,4)$ is the location of a local minimum.
- $D(\pm 2, 2) < 0$, so $(\pm 2, 2)$ are the locations of saddle points.

(b) We have

$$\nabla f(2,1) = \langle f_x(2,1), f_y(2,1)\rangle = \langle -12, 3\rangle = 3\langle -4, 1\rangle.$$

So, the unit vector in the steepest uphill direction from $(2,1)$ is

$$\frac{\nabla f(2,1)}{||\nabla f(2,1)||} = \frac{1}{\sqrt{17}}\langle -4, 1\rangle.$$

(c) We note that $\mathbf{r}(t) = \langle 0, -1\rangle$ when $t = 2$.
We have $\mathbf{r}'(t) = \langle -1, -t\rangle$, so $\mathbf{r}'(2) = \langle -1, -2\rangle$ and $||\mathbf{r}'(2)|| = \sqrt{5}$. Thus,

i. $\dfrac{df}{dt} = \nabla f(0,-1) \cdot \langle -1, -2\rangle = \langle 0, 15\rangle \cdot \langle -1, -2\rangle = -30$ meters per minute and

ii. $\dfrac{df}{ds} = \dfrac{\nabla f(0,-1) \cdot \langle -1, -2\rangle}{||\mathbf{r}'(2)||} = -\dfrac{30}{\sqrt{5}}$ meters per meter.

iii. Since these two derivatives are negative, we know the pudu is heading downhill.

2. Recall that the volume of a right circular cone is given by $V(r,h) = \frac{\pi}{3}r^2h$ where r is the radius and h is the height.

(a) (9 points) Determine the linear approximation of $V(r,h)$ centered at $(r,h) = (3,4)$.

(b) (4 points) Use your answer from (a) to approximate the volume of a right circular cone whose radius is 3.1 cm and height is 3.9 cm.

(c) (9 points) Use Taylor's Theorem to determine an upper bound on the error in approximating the volume of the cone with the linear approximation when $|r - 3| < 0.1$ and $|h - 4| < 0.1$. (For this problem only, you do not need to fully simplify your final answer, but it should be in a form that could be directly input into a simple calculator. To earn points on this problem you must use the techniques from class with Taylor's Theorem.)

(d) (9 points) Assume a cone has a radius of $r = 3$ cm and a height of $h = 4$ cm. It is exposed to some radioactive material and its dimensions start to change. If its volume remains unchanged but the height starts growing at a rate of 2 cm per hour, determine the rate of change of its radius. *Include the correct units in your final answer.*

Solution:

(a) We have the following first partial derivatives:

$$\frac{\partial V}{\partial r} = \frac{2\pi}{3}rh \qquad \text{and} \qquad \frac{\partial V}{\partial h} = \frac{\pi}{3}r^2.$$

At $(r,h) = (3,4)$, we have

$$V = 12\pi \qquad \text{and} \qquad \frac{\partial V}{\partial r} = 8\pi \qquad \text{and} \qquad \frac{\partial V}{\partial h} = 3\pi.$$

So, the linear approximation is given by

$$L(r,h) = 12\pi + 8\pi(r-3) + 3\pi(h-4).$$

(b)

$$V(3.1, 3.9) \approx L(3.1, 3.9) = 12\pi + \frac{8\pi}{10} - \frac{3\pi}{10} = \frac{25\pi}{2} \text{ cubic centimeters.}$$

(c) The second partial derivatives are

$$\frac{\partial^2 V}{\partial r^2} = \frac{2\pi}{3}h \qquad \text{and} \qquad \frac{\partial^2 V}{\partial r \partial h} = \frac{2\pi}{3}r \qquad \text{and} \qquad \frac{\partial^2 V}{\partial h^2} = 0.$$

The maximum values of each of these on $|r-3| < 0.1$ and $|h-4| < 0.1$ are

$$\left|\frac{\partial^2 V}{\partial r^2}\right| \le \frac{2\pi}{3}\left(\frac{41}{10}\right) = \frac{41\pi}{15} \qquad \text{and} \qquad \left|\frac{\partial^2 V}{\partial r \partial h}\right| \le \frac{2\pi}{3}\left(\frac{31}{10}\right) = \frac{31\pi}{15} \qquad \text{and} \qquad \left|\frac{\partial^2 V}{\partial h^2}\right| = 0.$$

Thus, an upper bound on the absolute value of the second partial derivatives on this region is $M = \frac{41\pi}{15}$. By Taylor's Theorem, the maximum possible error is

$$|E(x,y)| \le \frac{M}{(1+1)!}\left(\Delta r + \Delta h\right)^{1+1} = \frac{41\pi}{30}\left(\frac{1}{5}\right)^2 = \frac{41\pi}{750} \text{ cubic centimeters.}$$

(d) We need to find $\frac{dr}{dt}$. Using the chain rule, we have

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} \\ 0 &= 8\pi\frac{dr}{dt} + 3\pi(2) \\ \frac{dr}{dt} &= -\frac{3}{4} \text{ centimeters per hour.} \end{aligned}$$

3. The following two problems are not related.

(a) (9 points) Evaluate the following limit or show it does not exist: $\lim_{(x,y)\to(0,1)} \frac{xy}{x+(y-1)^2}$

(b) (9 points) Let $f(x,y)$ be defined by

$$f(x,y) = \begin{cases} \dfrac{(2x+4)^2 - (y-5)^2}{2x-y+9}, & (x,y) \neq (-4,1) \\ c, & (x,y) = (-4,1) \end{cases}$$

Is there a value of c that makes $f(x,y)$ continuous at $(-4,1)$? If so, find the value of c. If not, explain why not. (In either case be sure to justify your answer with the definition of continuity)

Solution:

(a) If we try to evaluate the function at the given point, we obtain $0/0$, an indeterminate form. So, more work is needed.

If we consider the path where $x = 0$, then as $(0,y) \to (0,1)$, we have

$$\frac{xy}{x+(y-1)^2} = \frac{0}{(y-1)^2} = 0 \to 0.$$

If we instead consider the path $y = 1$, then as $(x,1) \to (0,1)$, we have

$$\frac{xy}{x+(y-1)^2} = \frac{x}{x} = 1 \to 1.$$

Since the function approaches different values as (x,y) approaches $(0,1)$ along different paths, then the limit **does not exist.**

(b) We need $\lim_{(x,y)\to(-4,1)} f(x,y) = f(-4,1) = c$. So, we will evaluate the limit, which is a $0/0$-indeterminate form. Note that the numerator is the difference of two squares.

$$\begin{aligned}\lim_{(x,y)\to(-4,1)} \frac{(2x+4)^2-(y-5)^2}{2x-y+9} &= \lim_{(x,y)\to(-4,1)} \frac{(2x+4-y+5)(2x+4+y-5)}{2x-y+9}\\ &= \lim_{(x,y)\to(-4,1)} 2x+y-1\\ &= 2(-4)+1-1\\ &= -8.\end{aligned}$$

So, we need $c=-8$.

4. Consider the function $f(x,y) = e^{x^2+2x+y^2-4y}$.

   (a) (9 points) Use Lagrange Multipliers to determine both the **absolute maximum value** and **absolute minimum value** of $f(x,y)$ when $x^2+y^2=20$.

   (b) (3 points) Explain why the Extreme Value Theorem guarantees the existence of both absolute maximum and minimum values for $f(x,y)$ when $x^2+y^2 \le 20$.

   (c) (7 points) Determine the **absolute maximum value** and **absolute minimum value** of $f(x,y)$ when $x^2+y^2 \le 20$.

Solution:

(a) We see that

$$\frac{\partial f}{\partial x} = (2x+2)e^{x^2+2x+y^2-4y} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = (2y-4)e^{x^2+2x+y^2-4y}.$$

Our constraint is $g(x,y)=x^2+y^2=20$. Note that

$$\frac{\partial g}{\partial x} = 2x \qquad \text{and} \qquad \frac{\partial g}{\partial y} = 2y.$$

We need to know which (x,y)-values will solve

$$\begin{aligned}(2x+2)e^{x^2+2x+y^2-4y} &= 2\lambda x\\ (2y-4)e^{x^2+2x+y^2-4y} &= 2\lambda y\\ x^2+y^2 &= 20.\end{aligned}$$

If $x=0$, then the first equation would yield a contradiction. Likewise, if $y=0$, then the first equation would yield a contradiction.

We may assume x and y are nonzero. If we solve the first two equations for λ and equate, we have

$$\frac{2x+2}{2x}e^{x^2+2x+y^2-4y} = \frac{2y-4}{2y}e^{x^2+2x+y^2-4y},$$

which simplifies to $y=-2x$. If we apply this to the third equation, we have

$$20 = x^2+y^2 = x^2+(-2x)^2 = 5x^2,$$

which has solutions $x=\pm 2$. Since $y=-2x$, the two points to check are $(-2,4)$ and $(2,-4)$:

$$f(-2,4) = e^{4-4+16-16} = e^0 = 1 \qquad \text{and} \qquad f(2,-4) = e^{4+4+16+16} = e^{40}.$$

So, the absolute maximum and minimum values of f when $x^2+y^2=20$ are e^{40} and 1, respectively.

(b) Since $f(x, y)$ is **continuous** (because it is an exponential composed with a polynomial) on the given region and the given region is both **closed** and **bounded**, then the hypotheses of the Extreme Value Theorem are satisfied.

(c) First, we must find any critical points. But, $\frac{\partial f}{\partial x} = 0$ given $x = -1$ and $\frac{\partial f}{\partial y} = 0$ gives $y = 2$. So, the only critical point is $(-1, 2)$, which does lie in the given region since $(-1)^2 + 2^2 = 5 \leq 20$.

We see that

$$f(-1, 2) = e^{1-2+4-8} = e^{-5}.$$

Combined with the extrema on the boundary of the region, we see that the absolute maximum and minimum values of f when $x^2 + y^2 \leq 20$ are e^{40} and e^{-5}, respectively.