1. (14 pts) Consider the differential equation

$$x\frac{dy}{dx} = \frac{2x^2+1}{e^y}, \quad \text{where } x > 1.$$

Find the particular solution where $y(e) = 0$.

Solution: First, separate the variables and integrate to find the general solution:

$$\begin{aligned} e^y \frac{dy}{dx} &= \frac{2x^2+1}{x}\,dx = \left(2x + \frac{1}{x}\right)dx \\ \int e^y\,dy &= \int \left(2x + \frac{1}{x}\right)dx \\ e^y &= x^2 + \ln(x) + C \\ y &= \ln\left(x^2 + \ln(x) + C\right). \end{aligned}$$

Next, we solve for the particular value of C when $(x, y) = (e, o)$:

$$0 = \ln\left(e^2 + \ln(e) + C\right) \implies 1 = e^2 + 1 + C \implies C = -e^2.$$

Hence, the particular solution of the differential equation is

$$y = \ln\left(x^2 + \ln(x) - e^2\right).$$

2. (10 pts) A force of 3 pounds will stretch a spring $1/2$ feet beyond its natural length.

   (a) How far will an 8-pound force stretch the spring?

   (b) How much work does it take to stretch the spring to the distance you found in (a)?

On this problem, you may assume that the force F required to stretch a string x units beyond its natural length is given by $F = kx$, where k is a constant.

Solution:

   (a) First, we need to solve for the spring constant k. The given information says that $3 = k(1/2)$, so $k = 6$. Hence, an 8-pound force will yield the relation $8 = 6x$, which implies that $x = 4/3$ feet.

   (b) To determine the work involved in stretching the spring $4/3$ feet beyond its natural length, we integrate the force function:

$$W = \int_0^{4/3} 6x\,dx = \left(3x^2\right)\Big|_0^{4/3} = 3\left(\frac{4}{3}\right)^2 = \frac{48}{9}\text{foot-pounds.}$$

3. (24 pts) Consider the region $\mathcal{R}$ in Quadrant 1 bounded by $y = \sin(x)$ and $y = \cos(x)$ until their first point of intersection, and the y-axis.

   (a) Sketch the region, and label any intercepts and points of intersection.

For the following parts, set up but DO NOT evaluate integrals for the given quantities:

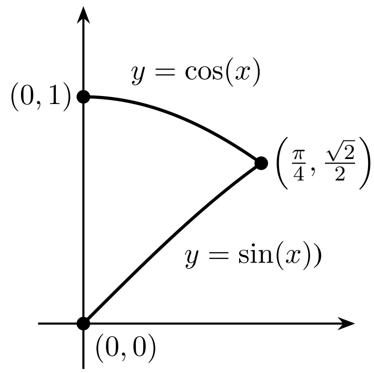
   (b) the volume of the solid generated by rotating $\mathcal{R}$ around the line $y = -1$;

   (c) the perimeter of the region;

   (d) the moment about the x-axis, assuming the region has density ρ.

Solution:

(a) Setting $\sin(x) = \cos(x)$ and solving for x gives the first intersection point in Quadrant 1 to be $(\pi/4, \sqrt{2}/2)$. Hence, the region is given by



(b)

$$V = \int_0^{\pi/4} \pi \left((\cos(x) + 1)^2 - (\sin(x) + 1)^2\right) \, dx$$

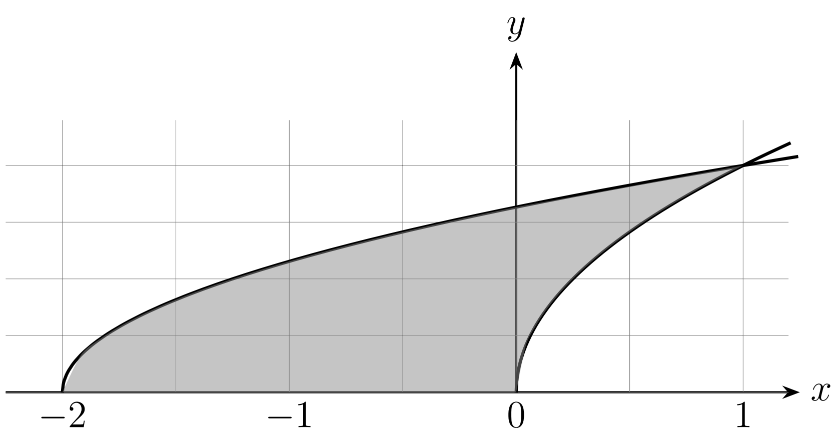
(c)

$$P = 1 + \int_0^{\pi/4} \sqrt{1 + \cos^2(x)} \, dx + \int_0^{\pi/4} \sqrt{1 + \sin^2(x)} \, dx$$

(d)

$$M_x = \int_0^{\pi/4} \frac{\rho}{2} \left(\cos^2(x) - \sin^2(x)\right) \, dx$$

4. (28 pts) Consider the region below, bounded by the curves $x = 3y^2 - 2$, $x = y^2$, and the x-axis.



(a) Compute the area of the region.

(b) Compute the volume of the solid generated by rotating the region around the line $y = 2$ using the shell method.

(c) Consider the portion of the curve $x = y^2$ which bounds the shaded region. Find the surface area of the shape formed by rotating the curve segment around the x-axis.

Solution:

(a)

$$\begin{aligned} A &= \int_0^1 (y^2 - 3y^2 + 2)\,dy \\ &= \int_0^1 (2 - 2y^2)\,dy \\ &= \left(2y - \frac{2}{3}y^3\right)\Big|_0^1 \\ &= 2 - \frac{2}{3} \\ &= \frac{4}{3}. \end{aligned}$$

(b)

$$\begin{aligned} V &= 2\pi \int_0^1 (2-y)(2-2y^2)\,dy \\ &= 2\pi \int_0^1 (4 - 4y^2 - 2y + 2y^3)\,dy \\ &= 2\pi \int_0^1 (2-y)(2-2y^2)\,dy \\ &= 2\pi \int_0^1 (4 - 2y - 4y^2 + 2y^3)\,dy \\ &= 2\pi \left(4y - y^2 - \frac{4}{3}y^3 + \frac{1}{2}y^4\right)\Big|_0^1 \\ &= 2\pi \left(4 - 1 - \frac{4}{3} + \frac{1}{2}\right) \\ &= \frac{13\pi}{3}. \end{aligned}$$

(c) Note that $x = y^2$ implies that $\frac{dx}{dy} = 2y$, so

$$\begin{aligned} SA &= 2\pi \int_0^1 y\sqrt{1 + (2y)^2}\,dy \\ &= 2\pi \int_0^1 y\sqrt{1 + 4y^2}\,dy \end{aligned}$$

using the substitution $u = 1 + 4y^2$, $du = 8y$ gives

$$\begin{aligned} &= \frac{\pi}{4}\int_1^5 \sqrt{u}\,du \\ &= \frac{\pi}{4}\left(\frac{2}{3}u^{3/2}\right)\Big|_1^5 \\ &= \frac{\pi}{6}\left(5^{3/2} - 1\right). \end{aligned}$$

5. (12 pts) Consider the series given by

$$\frac{1}{a} - \frac{1}{3a^2} + \frac{1}{9a^3} - \frac{1}{27a^4} + \cdots$$

(a) Write this series in $\sum$-notation.

(b) For which values of a will this series converge? Your answer can use inequalities or interval notation.

(c) Find the sum of the series when $a = \frac{1}{2}$.

Solution:

(a) Note that

$$\begin{aligned}\frac{1}{a} - \frac{1}{3a^2} + \frac{1}{9a^3} - \frac{1}{27a^4} + \cdots &= \frac{1}{a}\left(1 - \frac{1}{3a} + \frac{1}{9a^2} - \frac{1}{27a^3} + \cdots\right) \\ &= \frac{1}{a}\left(\sum_{n=0}^{\infty}\left(-\frac{1}{3a}\right)^n\right) \\ &= \sum_{n=0}^{\infty}\frac{1}{a}\left(-\frac{1}{3a}\right)^n,\end{aligned}$$

though there are many other valid ways of writing this in sigma notation.

(b) Note that this is a geometric series with $r = -\frac{1}{3a}$. Hence, it will converge when

$$\left|-\frac{1}{3a}\right| < 1 \implies \frac{1}{3|a|} < 1 \implies |a| > \frac{1}{3}.$$

In interval notation, this series will converge when a is in the interval $(-\infty, -1/3) \cup (1/3, \infty)$, and will diverge otherwise.

(c) When $a = 1/2$, the series becomes

$$\sum_{n=0}^{\infty} 2\left(-\frac{2}{3}\right)^n.$$

Using the geometric sum formula, the sum is given by

$$\sum_{n=0}^{\infty} 2\left(-\frac{2}{3}\right)^n = \frac{2}{1-(-2/3)} = \frac{6}{5}.$$

6. (12 pts) Determine whether the following quantities converge or diverge. If something converges, determine what it converges to. If something diverges, explain why. *Note: if you use L'Hôpital's Rule, state where you used it. Give the name of any other theorems you use.*

(a) the sequence $a_n = \dfrac{\ln(n+1)}{\sqrt{n}}$;

(b) the sequence $b_n = \dfrac{\sin(n)+1}{n+1}$;

(c) the series $\displaystyle\sum_{n=0}^{\infty} c_n$, which has partial sums given by $s_n = 2\left(\frac{1}{3}\right)^n$;

(d) the series $\sum d_n$, where the sequence $\{d_n\}$ is given by $d_n = \left(2 - \dfrac{1}{n}\right)^2$.

Solution:

(a) This sequence converges:

$$\lim_{n\to\infty}\frac{\ln(n+1)}{\sqrt{n}} \overset{LH}{=} \lim_{n\to\infty}\frac{1/(n+1)}{1/(2\sqrt{n})} = \lim_{n\to\infty}\frac{2\sqrt{n}}{n+1} \overset{LH}{=} \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0.$$

(b) This sequence converges. Note that $-1 \leq \sin(n) \leq 1$, so $0 \leq \sin(n) + 1 \leq 2$, so

$$0 \leq \frac{\sin(n)+1}{n+1} \leq \frac{2}{n+1}.$$

Since $\lim_{n\to\infty} \frac{2}{n+1} = 0$, the Squeeze Theorem says that this sequence also converges to 0.

(c) This series converges to 0, since

$$\sum_{n=0}^{\infty} c_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} 2\left(\frac{1}{3}\right)^n = 0.$$

(d) This series diverges by the Test for Divergence since the terms don't go to 0:

$$\lim_{n\to\infty} \left(2 - \frac{1}{n}\right)^2 = (2-0)^2 = 4.$$