1. [2360/030426 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a two-column table (letter - answer) completely separate from any work you do to arrive at the answer.

   (a) If $|\mathbf{A}| = 0$, then the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ is inconsistent.

   (b) If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times m$ matrix and $\mathbf{BA}$ does not have 0 as an eigenvalue, then $(\mathbf{BA})^{-1}$ exists.

   (c) The Wronskian of the functions $\{1 + t - t^2, t^2, 1\}$ is 0.

   (d) If the system of equations $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$, $\mathbf{A}$ an $n \times n$ matrix, is inconsistent for some $\vec{\mathbf{b}} \in \mathbb{R}^n$, then $\mathbf{A}$ is singular.

   (e) If $\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 7 & 9 \end{bmatrix}$, then $\mathbf{A}^3\mathbf{A}^{\mathrm{T}}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a solution for all $\vec{\mathbf{b}} \in \mathbb{R}^2$.

   SOLUTION:

   (a) **FALSE** Homogeneous systems are always consistent since they possess the trivial solution ($\vec{\mathbf{x}} = \vec{\mathbf{0}}$) at a minimum.

   (b) **TRUE** Since $\mathbf{BA}$ is $n \times n$ with all eigenvalues nonzero, it is invertible.

   (c) **FALSE**

   $$W(t) = \begin{vmatrix} 1+t-t^2 & t^2 & 1 \\ 1-2t & 2t & 0 \\ -2 & 2 & 0 \end{vmatrix} = 1(-1)^{1+3} \begin{vmatrix} 1-2t & 2t \\ -2 & 2 \end{vmatrix} = 2 - 4t - (-4t) = 2$$

   (d) **TRUE** If the system were consistent for all $\vec{\mathbf{b}} \in \mathbb{R}^n$, then $\vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$.

   (e) **TRUE** $|\mathbf{A}| = -26 \neq 0 \implies |\mathbf{A}^{\mathrm{T}}| \neq 0$. Thus both $\mathbf{A}$ and $\mathbf{A}^{\mathrm{T}}$ are invertible. Also, $\mathbf{A}^3$ is invertible with $(\mathbf{A}^3)^{-1} = (\mathbf{A}^{-1})^3$. Hence

   $$\left(\mathbf{A}^3\mathbf{A}^{\mathrm{T}}\right)^{-1}\left(\mathbf{A}^3\mathbf{A}^{\mathrm{T}}\right)\vec{\mathbf{x}} = \left(\mathbf{A}^3\mathbf{A}^{\mathrm{T}}\right)^{-1}\vec{\mathbf{b}}$$
   $$\vec{\mathbf{x}} = \left(\mathbf{A}^{\mathrm{T}}\right)^{-1}\left(\mathbf{A}^3\right)^{-1}\vec{\mathbf{b}} = \left(\mathbf{A}^{-1}\right)^{\mathrm{T}}\left(\mathbf{A}^{-1}\right)^3\vec{\mathbf{b}}$$

   ■

2. [2360/030426 (15 pts)] In certain models of the economy involving the interdependencies of various sectors, solutions to matrix equations of the form $\vec{\mathbf{x}} = \mathbf{T}\vec{\mathbf{x}} + \vec{\mathbf{d}}$ are required. This equation can be written in the equivalent form $(\mathbf{I} - \mathbf{T})\vec{\mathbf{x}} = \vec{\mathbf{d}}$. If $\mathbf{T} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ and $\vec{\mathbf{d}} = \begin{bmatrix} 10 & 20 & 30 \end{bmatrix}^{\mathrm{T}}$ use an appropriate inverse matrix to find $\vec{\mathbf{x}}$. You must find this inverse using Gauss-Jordan elimination/reduction with zero points awarded for using any other method. Hint: There are no fractions in this problem.

   SOLUTION:

   The solution we seek is $\vec{\mathbf{x}} = (\mathbf{I} - \mathbf{T})^{-1}\vec{\mathbf{d}}$ where $\mathbf{I} - \mathbf{T} = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -2 & -1 \\ -2 & -1 & -1 \end{bmatrix}$.

   $$\left[\begin{array}{ccc|ccc} -1 & -1 & -1 & 1 & 0 & 0 \\ -1 & -2 & -1 & 0 & 1 & 0 \\ -2 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3^*=-2R_1+R_3]{R_2^*=-1R_1+R_2} \left[\begin{array}{ccc|ccc} -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow[R_3^*=1R_2+R_3]{R_1^*=-1R_2+R_1}$$

   $$\left[\begin{array}{ccc|ccc} -1 & 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{array}\right] \xrightarrow{R_1^*=1R_3+R_1} \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{array}\right] \xrightarrow{R_1^*=-1R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{array}\right]$$

   Thus $(\mathbf{I} - \mathbf{T})^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ -3 & 1 & 1 \end{bmatrix}$ and

   $$\vec{\mathbf{x}} = (\mathbf{I} - \mathbf{T})^{-1}\vec{\mathbf{d}} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ -3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} = \begin{bmatrix} -20 \\ -10 \\ 20 \end{bmatrix}$$

   ■

3. [2360/030426 (12 pts)] For each of the following subsets, $\mathbb{W}$, of the given vector space, $\mathbb{V}$, determine if $\mathbb{W}$ is a subspace. If it is a subspace, simply write **YES**. If it is not a subspace, write **NO** along with the Roman numerals of all the axioms that fail to hold. Assume the standard operations in each vector space. One point awarded for correct YES/NO answer, one point for correct Roman numeral(s). No other partial credit available and no work need be shown.

I. The set is not closed under vector addition   II. The set is not closed under scalar multiplication

(a) $\mathbb{V} = \mathbb{R}^2;\ \mathbb{W} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \,\middle|\, a \in \mathbb{Q} \text{ (rational numbers)};\ b \in \mathbb{R} \right\}$

(b) $\mathbb{V} = \mathbb{M}_{n\times n};\ \mathbb{W} = \left\{ \mathbf{A} \in \mathbb{M}_{n\times n} \,\middle|\, \mathbf{A}^{\mathrm{T}} \text{ is singular} \right\}$

(c) $\mathbb{V} = \mathbb{P}_2;\ \mathbb{W} = \left\{ p(t) \,\middle|\, p(t) = 1 + a_1 t + a_2 t^2;\ a_1, a_2 \in \mathbb{R} \right\}$

(d) $\mathbb{V} = \mathcal{C}^1(-1,1);\ \mathbb{W} = \left\{ f(t) \,\middle|\, f'(t) \geq 0 \right\}$

(e) $\mathbb{V} = \mathbb{R}^3;\ \mathbb{W} = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \,\middle|\, ab = 0; a, b, c \in \mathbb{R}, \right\}$

(f) $\mathbb{V} = \mathbb{M}_{2\times 2};\ \mathbb{W} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a + b + c + d = 0; a, b, c, d \in \mathbb{R}, \right\}$

SOLUTION:

(a) **NO** - II. For example, $\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \in \mathbb{W}$ but $\pi \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix} \notin \mathbb{W}$

(b) **NO** - I. For example, $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Since both $\mathbf{A}^{\mathrm{T}}$ and $\mathbf{B}^{\mathrm{T}}$ are singular, $\mathbf{A}, \mathbf{B} \in \mathbb{W}$. However, $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the transpose of which is nonsingular, implying $\mathbf{A} + \mathbf{B} \notin \mathbb{W}$.

(c) **NO** - I, II. For example, $p(t) = 1 + 2t \in \mathbb{W}$. But $2p(t) = 2 + 4t \notin \mathbb{W}$. Also, $p(t) + p(t) = 2 + 4t \notin \mathbb{W}$.

(d) **NO** - II. For example $f(t) = t \in \mathbb{W}$ but $-1f(t) = -t \notin \mathbb{W}$

(e) **NO** - I. For example, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{W}$ but $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \notin \mathbb{W}$

(f) **YES**

■

4. [2360/030426 (15 pts)] Let $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{\mathbf{v}}_2 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$. Justify your answers to the following questions.

(a) (5 pts) Is $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$ a linearly dependent set of vectors?

(b) (5 pts) If $\vec{\mathbf{v}}_3 = \begin{bmatrix} -5 \\ -11 \\ 20 \end{bmatrix}$, is $\vec{\mathbf{v}}_3 \in \operatorname{span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$? Your justification must use matrices for full credit.

(c) (5 pts) If $\vec{\mathbf{v}}_4 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$, find $\operatorname{span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_4\}$.

SOLUTION:

(a) No. Here's why. Can we find c_1, c_2, not both 0, such that $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 = \vec{\mathbf{0}}$?

$$c_1\begin{bmatrix}1\\1\\2\end{bmatrix} + c_2\begin{bmatrix}2\\3\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \iff \begin{bmatrix}1 & 2\\1 & 3\\2 & -1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1 & 2 & 0\\1 & 3 & 0\\2 & -1 & 0\end{array}\right] \to \left[\begin{array}{cc|c}1 & 2 & 0\\0 & 1 & 0\\0 & -5 & 0\end{array}\right] \to \left[\begin{array}{cc|c}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{array}\right]$$

The only solution is $c_1 = c_2 = 0$, implying that the set of vectors is linearly independent, not linearly dependent.

(b) Now we need to find see if c_1, c_2 exist such that $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_3$.

$$c_1\begin{bmatrix}1\\1\\2\end{bmatrix} + c_2\begin{bmatrix}2\\3\\-1\end{bmatrix} = \begin{bmatrix}-5\\-11\\20\end{bmatrix} \iff \begin{bmatrix}1 & 2\\1 & 3\\2 & -1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}-5\\-11\\20\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1 & 2 & -5\\1 & 3 & -11\\2 & -1 & 20\end{array}\right] \to \left[\begin{array}{cc|c}1 & 2 & -5\\0 & 1 & -6\\0 & -5 & 30\end{array}\right] \to \left[\begin{array}{cc|c}1 & 0 & 7\\0 & 1 & -6\\0 & 0 & 0\end{array}\right]$$

Thus, $c_1 = 7, c_2 = -6$ so that $\vec{\mathbf{v}}_3 = 7\vec{\mathbf{v}}_1 - 6\vec{\mathbf{v}}_2$ showing that $\vec{\mathbf{v}}_3 \in \text{span}\,\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$.

(c) Now we look at $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 = \vec{\mathbf{v}}$ where $\vec{\mathbf{v}} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$ is an arbitrary vector in $\mathbb{R}^3$.

$$c_1\begin{bmatrix}1\\1\\2\end{bmatrix} + c_2\begin{bmatrix}2\\3\\-1\end{bmatrix} + c_3\begin{bmatrix}0\\0\\-1\end{bmatrix} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} \iff \begin{bmatrix}1 & 2 & 0\\1 & 3 & 0\\2 & -1 & -1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$$

Since

$$\begin{vmatrix}1 & 2 & 0\\1 & 3 & 0\\2 & -1 & -1\end{vmatrix} = (-1)(-1)^{3+3}\begin{vmatrix}1 & 2\\1 & 3\end{vmatrix} = -1 \neq 0$$

the coefficient matrix is invertible, implying that unique c_1, c_2, c_3 exist regardless of $\vec{\mathbf{v}}$. Thus span $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_4\} = \mathbb{R}^3$.

Alternatively, if $\vec{\mathbf{v}} = \vec{\mathbf{0}}$, the nonvanishing determinant would indicate that the three vectors are linearly independent. Since the dimension of $\mathbb{R}^3$ is three these vectors form a basis for $\mathbb{R}^3$.

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5. [2360/030426 (17 pts)] Let $\mathbf{A} = \begin{bmatrix}1 & -2 & 0\\2 & 1 & 0\\0 & 0 & -1\end{bmatrix}$.

(a) (6 pts) Find the eigenvalues of $\mathbf{A}$ and state their algebraic multiplicities.

(b) (5 pts) Find a basis for the eigenspace corresponding to the real eigenvalue of $\mathbf{A}$. What is the geometric multiplicity of the real eigenvalue?

(c) (6 pts) Let $\vec{\mathbf{v}} = \begin{bmatrix}1\\-i\\0\end{bmatrix}$.

i. (2 pts) Compute $\mathbf{A}\vec{\mathbf{v}}$.

ii. (2 pts) Compute $(1+2i)\vec{\mathbf{v}}$.

iii. (2 pts) Why are your answers to (i) and (ii) the same?

SOLUTION:

(a)

$$|\mathbf{A}-\lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & -2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = (-1-\lambda)(-1)^{3+3}\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = (-1-\lambda)\left[(1-\lambda)^2+4\right]$$

$$= (-1-\lambda)\left(\lambda^2-2\lambda+5\right) = 0$$

Clearly $\lambda = -1$ is one of the eigenvalues. From the quadratic formula the others are

$$\lambda = \frac{2 \pm \sqrt{(-2)^2-4(1)(5)}}{2} = \frac{2\pm\sqrt{-16}}{2} = \frac{2\pm 4i}{2} = 1 \pm 2i$$

The eigenvalues are $-1, 1+2i, 1-2i$, with each having algebraic multiplicity of 1.

(b) We solve $[\mathbf{A}-(-1)\mathbf{I}]\,\vec{\mathbf{v}} = (\mathbf{A}+\mathbf{I})\,\vec{\mathbf{v}} = \mathbf{0}$:

$$\left[\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

A basis for the eigenspace is $\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$. The geometric multiplicity of the eigenvalue $\lambda = -1$ is 1, the dimension of the eigenspace.

(c) i.

$$\mathbf{A}\vec{\mathbf{v}} = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} = \begin{bmatrix} 1+2i \\ 2-i \\ 0 \end{bmatrix}$$

ii.

$$(1+2i)\begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} = \begin{bmatrix} 1+2i \\ 2-i \\ 0 \end{bmatrix}$$

iii. The above two results are equal because $1+2i$ is an eigenvalue of $\mathbf{A}$ with associated eigenvector $\vec{\mathbf{v}}$.

■

6. [2360/030426 (16 pts)] Given the matrices,

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 2 & -6 \\ 4 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 3 & -5 & 0 & 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -2 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} i & 1-i \\ 1+i & -i \end{bmatrix}$$

calculate the following or explain why it cannot be calculated. Simplify your answers.

(a) $\mathbf{B}+\mathbf{C}^{\mathrm{T}}$ (b) $i\mathbf{D}$ (c) $|2\mathbf{B}|$ (d) $\mathrm{Tr}\left(\mathbf{C}^{\mathrm{T}}\mathbf{C}\right)$ (e) $\mathbf{AC}$ (f) $\mathbf{B}^2$ (g) $|\mathbf{A}|$ (h) The RREF of $\mathbf{C}$

SOLUTION:

(a)

$$\mathbf{B}+\mathbf{C}^{\mathrm{T}} = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 3 & -5 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 0 & 1 \\ 3 & -5 & 0 & -2 \end{bmatrix}$$

(b)

$$i\mathbf{D} = i\begin{bmatrix} i & 1-i \\ 1+i & -i \end{bmatrix} = \begin{bmatrix} i^2 & 1i-i^2 \\ 1i+i^2 & -i^2 \end{bmatrix} = \begin{bmatrix} -1 & 1+i \\ -1+i & 1 \end{bmatrix}$$

(c) $2\mathbf{B}$ is not square

(d)

$$\mathbf{C}^{\mathrm{T}}\mathbf{C} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \implies \mathrm{Tr}\left(\mathbf{C}^{\mathrm{T}}\mathbf{C}\right) = 8$$

(e)

$$\mathbf{AC} = \begin{bmatrix} 1 & 5 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 2 & -6 \\ 4 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 4 & 0 \\ 6 & 12 \\ 8 & 0 \end{bmatrix}$$

(f) $(2 \times 4)(2 \times 4)$ incompatible

(g)

$$|\mathbf{A}| = 5(-1)^{1+2} \begin{vmatrix} 2 & 1 & 0 \\ 3 & 2 & -6 \\ 4 & 1 & 0 \end{vmatrix} = -5\left[(-6)(-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix}\right] = -5(6)(-2) = 60$$

(h)

$$\text{RREF of } \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

■

7. [2360/030426 (15 pts)] Consider the linear system $\begin{array}{r} 2x_1 + 4x_2 + 5x_3 + 9x_4 = 9 \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 6 \end{array}$.

(a) (1 pt) Is the system overdetermined or underdetermined?

(b) (5 pts) Find a particular solution for the system.

(c) (5 pts) Find a basis and the dimension of the solution space of the associated homogeneous system.

(d) (4 pts) Find the general solution of the system by applying the Nonhomogeneous Principle for linear equations.

SOLUTION:

(a) Underdetermined - more variables than equations

(b)

$$\left[\begin{array}{cccc|c} 2 & 4 & 5 & 9 & 9 \\ 1 & 2 & 3 & 4 & 6 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 6 \\ 2 & 4 & 5 & 9 & 9 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 6 \\ 0 & 0 & -1 & 1 & -3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 0 & 7 & -3 \\ 0 & 0 & 1 & -1 & 3 \end{array}\right]$$

Free variables are $x_2 = r$ and $x_4 = s$. Then $x_1 = -3 - 2r - 7s$ and $x_3 = 3 + s$. Setting $r = s = 0$ a particular solution is

$$\vec{\mathbf{x}}_p = \begin{bmatrix} -3 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

(c) The solution to the associated homogeneous problem is

$$\vec{\mathbf{x}}_h = \begin{bmatrix} -2r - 7s \\ r \\ s \\ s \end{bmatrix} = r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -7 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \ r, s \in \mathbb{R}$$

A basis for the solution space, which has dimension 2, is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(d)

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p = r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -7 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 3 \\ 0 \end{bmatrix} \quad r, s \in \mathbb{R}$$

■