- This exam is worth 100 points and has 7 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"× 11" crib sheet with writing on one side.

0. At the top of the page containing your solution to problem 1, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/030426 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a two-column table (letter - answer) completely separate from any work you do to arrive at the answer.

   (a) If $|\mathbf{A}| = 0$, then the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ is inconsistent.

   (b) If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times m$ matrix and $\mathbf{BA}$ does not have 0 as an eigenvalue, then $(\mathbf{BA})^{-1}$ exists.

   (c) The Wronskian of the functions $\left\{1 + t - t^2, t^2, 1\right\}$ is 0.

   (d) If the system of equations $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}},\ \mathbf{A}$ an $n \times n$ matrix, is inconsistent for some $\vec{\mathbf{b}} \in \mathbb{R}^n$, then $\mathbf{A}$ is singular.

   (e) If $\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 7 & 9 \end{bmatrix}$, then $\mathbf{A}^3\mathbf{A}^{\mathrm{T}}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a solution for all $\vec{\mathbf{b}} \in \mathbb{R}^2$.

2. [2360/030426 (15 pts)] In certain models of the economy involving the interdependencies of various sectors, solutions to matrix equations of the form $\vec{\mathbf{x}} = \mathbf{T}\vec{\mathbf{x}} + \vec{\mathbf{d}}$ are required. This equation can be written in the equivalent form $(\mathbf{I} - \mathbf{T})\,\vec{\mathbf{x}} = \vec{\mathbf{d}}$. If $\mathbf{T} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ and $\vec{\mathbf{d}} = \begin{bmatrix} 10 & 20 & 30 \end{bmatrix}^{\mathrm{T}}$ use an appropriate inverse matrix to find $\vec{\mathbf{x}}$. You must find this inverse using Gauss-Jordan elimination/reduction with zero points awarded for using any other method. Hint: There are no fractions in this problem.

3. [2360/030426 (12 pts)] For each of the following subsets, $\mathbb{W}$, of the given vector space, $\mathbb{V}$, determine if $\mathbb{W}$ is a subspace. If it is a subspace, simply write **YES**. If it is not a subspace, write **NO** along with the Roman numerals of all the axioms that fail to hold. Assume the standard operations in each vector space. One point awarded for correct YES/NO answer, one point for correct Roman numeral(s). No other partial credit available and no work need be shown.

   I. The set is not closed under vector addition   II. The set is not closed under scalar multiplication

   (a) $\mathbb{V} = \mathbb{R}^2;\ \mathbb{W} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \,\middle|\, a \in \mathbb{Q} \text{ (rational numbers)};\ b \in \mathbb{R} \right\}$

   (b) $\mathbb{V} = \mathbb{M}_{n\times n};\ \mathbb{W} = \left\{ \mathbf{A} \in \mathbb{M}_{n\times n} \,\middle|\, \mathbf{A}^{\mathrm{T}} \text{ is singular} \right\}$

   (c) $\mathbb{V} = \mathbb{P}_2;\ \mathbb{W} = \left\{ p(t) \,\middle|\, p(t) = 1 + a_1 t + a_2 t^2;\ a_1, a_2 \in \mathbb{R} \right\}$

   (d) $\mathbb{V} = \mathcal{C}^1(-1, 1);\ \mathbb{W} = \left\{ f(t) \,\middle|\, f'(t) \geq 0 \right\}$

   (e) $\mathbb{V} = \mathbb{R}^3;\ \mathbb{W} = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \,\middle|\, ab = 0; a, b, c \in \mathbb{R}, \right\}$

   (f) $\mathbb{V} = \mathbb{M}_{2\times 2};\ \mathbb{W} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a + b + c + d = 0; a, b, c, d \in \mathbb{R}, \right\}$

MORE PROBLEMS BELOW/ON REVERSE

4. [2360/030426 (15 pts)] Let $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{\mathbf{v}}_2 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$. Justify your answers to the following questions.

(a) (5 pts) Is $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$ a linearly dependent set of vectors?

(b) (5 pts) If $\vec{\mathbf{v}}_3 = \begin{bmatrix} -5 \\ -11 \\ 20 \end{bmatrix}$, is $\vec{\mathbf{v}}_3 \in \text{span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$? Your justification must use matrices for full credit.

(c) (5 pts) If $\vec{\mathbf{v}}_4 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$, find $\text{span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_4\}$.

5. [2360/030426 (17 pts)] Let $\mathbf{A} = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(a) (6 pts) Find the eigenvalues of $\mathbf{A}$ and state their algebraic multiplicities.

(b) (5 pts) Find a basis for the eigenspace corresponding to the real eigenvalue of $\mathbf{A}$. What is the geometric multiplicity of the real eigenvalue?

(c) (6 pts) Let $\vec{\mathbf{v}} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$.

i. (2 pts) Compute $\mathbf{A}\vec{\mathbf{v}}$.

ii. (2 pts) Compute $(1+2i)\vec{\mathbf{v}}$.

iii. (2 pts) Why are your answers to (i) and (ii) the same?

6. [2360/030426 (16 pts)] Given the matrices,

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 2 & -6 \\ 4 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 3 & -5 & 0 & 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -2 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} i & 1-i \\ 1+i & -i \end{bmatrix}$$

calculate the following or explain why it cannot be calculated. Simplify your answers.

(a) $\mathbf{B} + \mathbf{C}^{\mathrm{T}}$ (b) $i\mathbf{D}$ (c) $|2\mathbf{B}|$ (d) $\mathrm{Tr}\left(\mathbf{C}^{\mathrm{T}}\mathbf{C}\right)$ (e) $\mathbf{AC}$ (f) $\mathbf{B}^2$ (g) $|\mathbf{A}|$ (h) The RREF of $\mathbf{C}$

7. [2360/030426 (15 pts)] Consider the linear system $\begin{aligned} 2x_1 + 4x_2 + 5x_3 + 9x_4 &= 9 \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= 6 \end{aligned}$.

(a) (1 pt) Is the system overdetermined or underdetermined?

(b) (5 pts) Find a particular solution for the system.

(c) (5 pts) Find a basis and the dimension of the solution space of the associated homogeneous system.

(d) (4 pts) Find the general solution of the system by applying the Nonhomogeneous Principle for linear equations.