

1. (30 points) The following parts are not related. You may use any method to find the derivatives.

(a) Find  $f'(x)$  given  $f(x) = \frac{1}{x} - 2x^3 \sec(x)$ . Leave your derivative unsimplified.

(b) Find  $g'(x)$  given  $g(x) = \frac{\sin(x^3 + \sqrt{x})}{x^2 - 10x}$ . Leave your derivative unsimplified.

(c) Consider the curve given by:  $Ax^2 + By^2 = 40$ , where  $A$  and  $B$  are non-zero constants.

(i) Find  $\frac{dy}{dx}$ .

(ii) Find  $\frac{d^2y}{dx^2}$ .

(iii) Let  $A = 10$  and let  $B = 10$ . Where is/are there a horizontal tangent line(s)? Specify the  $(x, y)$  coordinate(s).

**Solution:**

(a) We will find this derivative using the quotient rule for the first term and the product rule for the second term.

$$\begin{aligned} f'(x) &= \frac{x \cdot 0 - 1 \cdot 1}{x^2} - 2(x^3 \cdot \sec(x) \tan(x) + \sec(x) \cdot 3x^2) \\ &= -\frac{1}{x^2} - 2x^3 \sec(x) \tan(x) - 6x^2 \sec(x) \end{aligned}$$

Therefore, 
$$f'(x) = -\frac{1}{x^2} - 2x^3 \sec(x) \tan(x) - 6x^2 \sec(x)$$

(b) We will find this derivative using the quotient rule and the chain rule.

Therefore, 
$$g'(x) = \frac{(x^2 - 10x)(\cos(x^3 + \sqrt{x}) \cdot (3x^2 + \frac{1}{2\sqrt{x}}) - (\sin(x^3 + \sqrt{x})) \cdot (2x - 10))}{(x^2 - 10x)^2}$$

(c) (i) We use implicit differentiation to find  $dy/dx$ .

$$\frac{d}{dx} (Ax^2 + By^2) = \frac{d}{dx} (40)$$

$$2Ax + 2By \frac{dy}{dx} = 0$$

$$2By \frac{dy}{dx} = -2Ax$$

$$By \frac{dy}{dx} = -Ax$$

$$\frac{dy}{dx} = -\frac{Ax}{By}$$

Therefore, 
$$\frac{dy}{dx} = -\frac{Ax}{By}$$

(ii) We will use implicit differentiation and the quotient rule to find  $\frac{d^2y}{dx^2}$ .

$$\begin{aligned}\frac{d}{dx} \left( \frac{dy}{dx} \right) &= \frac{d}{dx} \left( -\frac{Ax}{By} \right) \\ &= \frac{(By) \cdot (-A) - (-Ax) \cdot (B \frac{dy}{dx})}{B^2y^2} \\ &= \frac{-ABy + ABx \frac{dy}{dx}}{B^2y^2} \\ &= \frac{-ABy + ABx \left( -\frac{Ax}{By} \right)}{B^2y^2} \\ &= \frac{-ABy - \frac{A^2x^2}{y}}{B^2y^2} \\ &= \frac{-ABy^2 - A^2x^2}{B^2y^3}\end{aligned}$$

Therefore,  $\boxed{\frac{d^2y}{dx^2} = \frac{-ABy^2 - A^2x^2}{B^2y^3}}$ .

(iii) Let  $A = 10$  and let  $B = 10$ . To determine the horizontal tangent lines, we will let  $\frac{dy}{dx} = 0$  and solve for  $x$  and  $y$ .

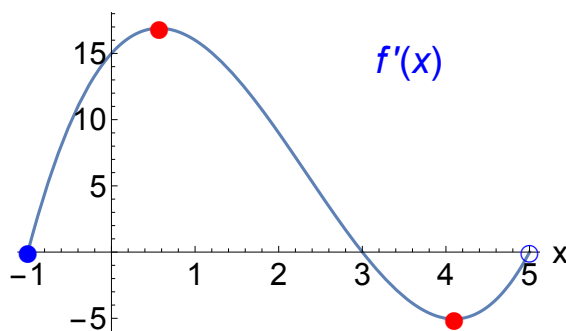
$$-\frac{10x}{10y} = 0 \implies x = 0$$

If  $x = 0$ , then the given relation becomes:  $0 + 10y^2 = 40$

$$\implies y^2 = 4 \implies y = \pm 2$$

Therefore, the locations of the horizontal tangents are  $\boxed{(0, -2), (0, 2)}$ .

2. (21 points) Answer the following questions about the function  $f(x)$ , whose domain is  $[-1, 5]$ . Please note that the graph of its **derivative**,  $y = f'(x)$ , is shown below. The derivative  $f'(x)$  attains its maximal value at  $(0.57, 16.9)$  and its minimal value at  $(4.1, -5.0)$ . The graph goes through the points  $(2, 10)$  and  $(3, 0)$ . *Note: While justification is not strictly necessary, providing it for your answers will significantly aid your solution construction.*



- (a) For what value(s) of  $x$  does  $y = f(x)$  have **horizontal tangents**?
- (b) On what open interval(s) does  $y = f(x)$  have **positive slope**?
- (c) On what open interval(s) does  $y = f(x)$  have **negative slope**?
- (d) Find the exact value of

$$L = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}.$$

**Solution:**

- (a) Horizontal tangents occur where  $f'(x) = 0$  and  $x$  is in the domain. The zeros of  $f'(x)$  are  $x = -1, 3, 5$ , but  $5 \notin [-1, 5]$ . Therefore  $y = f(x)$  have horizontal tangents at

$$\boxed{x = -1, 3}$$

- (b)  $f(x)$  has positive slope where  $f'(x) > 0$ . Testing intervals gives  $f'(x) > 0$  on  $(-1, 3)$ .

$$\boxed{(-1, 3)}$$

- (c)  $f(x)$  has negative slope where  $f'(x) < 0$ . This occurs on  $(3, 5)$ .

$$\boxed{(3, 5)}$$

- (d) By definition of derivative,

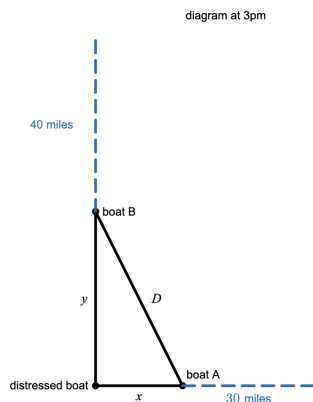
$$L = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} = f'(2) = 10$$

$$\boxed{L = 10}$$

3. (12 points) A call to the coast guard was received just before 2 pm from a distressed boat. One coast guard boat, boat A, is 50 miles east of the distressed boat and starts **moving west** at 2 pm at a rate of 30 miles per hour. Another coast guard boat, boat B, is 80 miles north and starts **moving south** at 2 pm at a rate of 40 miles per hour. At what rate is the distance between boat A and boat B changing at **3 pm**?

**Solution:**

First we set up a diagram to illustrate the situation at 3pm (after both coast guard boats have been traveling for 1 hour). From 2pm to 3pm boat A, which is moving at a speed of 30 miles per hour, has traveled 30 miles. From 2pm to 3pm boat B, which is moving at a speed of 40 miles per hour, has traveled 40 miles. These distances have been labeled in the diagram.



We are **given** the rate of boat A and we express this as a rate of change of the distance  $x$  from the distressed boat,  $\frac{dx}{dt} = -30$ . The rate is negative because the coast guard boat is approaching the distressed boat and the distance  $x$  is decreasing with time.

We are **given** the rate of boat B and we express this as a rate of change of the distance  $y$  from the distressed boat,  $\frac{dy}{dt} = -40$ . The rate is negative because the coast guard boat is approaching the distressed boat and the distance  $y$  is decreasing with time.

Because the boats have been traveling for 1 hour, we know  $x = 20$  and  $y = 40$  at 3pm.

We are **finding** the rate of change of the distance between the two boats at 3pm. If this distance is denoted as  $D$  then we are trying to find  $\left. \frac{dD}{dt} \right|_{3pm}$ .

We set up an equation that relates the given information to what we are trying to find by utilizing the Pythagorean theorem:

$$D^2 = x^2 + y^2$$

Implicitly differentiating with respect to time we get:

$$2D \frac{dD}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

Solving for  $\frac{dD}{dt}$  we get:

$$\frac{dD}{dt} = \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{D}$$

We do not know  $D$  at 3pm so we use the Pythagorean theorem  $D^2 = x^2 + y^2$ :

$$D^2 = (20)^2 + (40)^2$$

$$D^2 = 400 + 1600$$

$$D^2 = 2000$$

$$D = \sqrt{2000}$$

$$D = 20\sqrt{5}$$

We now answer the problem:

$$\begin{aligned} \left. \frac{dD}{dt} \right|_{3pm} &= \frac{20(-30) + 40(-40)}{20\sqrt{5}} \\ &= \frac{-600 - 1600}{20\sqrt{5}} \\ &= \frac{-2200}{20\sqrt{5}} \\ &= \boxed{-\frac{110}{\sqrt{5}} \text{ miles/hour}} \end{aligned}$$

4. (21 points) Consider the function  $f$  in the interval  $[0, 2\pi]$  where

$$f(x) = 2x + 2 \cos x - 1$$

- Determine all critical numbers of  $f(x)$  in the interval  $[0, 2\pi]$ .
- Find the absolute maximum and absolute minimum values of  $f(x)$  in the interval  $[0, 2\pi]$ . Clearly indicate the  $x$ -coordinates where these occur.
- Use the Intermediate Value Theorem to show that  $2x + 2 \cos x - 1 = -\frac{1}{2}$  has at least one solution. First state the hypotheses of the intermediate value theorem and confirm that they are satisfied. Then determine an interval  $[a, b]$  where the solution can be found. Note: Your  $a$  and  $b$  may be different than the interval specified in part (a).

**Solution:**

- The critical numbers are obtained by setting

$$\begin{aligned} f'(x) &= 0 \\ 2 - 2 \sin(x) &= 0 \\ \sin x &= 1 \\ x &= \boxed{\frac{\pi}{2}} \end{aligned}$$

- In order to find the absolute minimum and maximum values, we evaluate the function at the critical point as well as the endpoints. We have,

$$\begin{aligned} f(0) &= 1 \\ f(2\pi) &= 4\pi + 1 \\ f\left(\frac{\pi}{2}\right) &= \pi - 1 \end{aligned}$$

Hence the absolute maximum value is  $\boxed{4\pi + 1 \text{ at } x = 2\pi}$  and the absolute minimum value is  $\boxed{1 \text{ at } x = 0}$

- We construct the function  $g(x) = (2x + 2 \cos x - 1) - \left(-\frac{1}{2}\right) = 2x + 2 \cos x - \frac{1}{2}$ . Now,  $g(x)$  is made out of a polynomial function  $2x - \frac{1}{2}$  and a trigonometric function  $2 \cos x$ , both of which are continuous for all real numbers  $x$ . Hence  $g(x)$  is continuous for all real numbers. Let us pick an interval  $[-\pi, 0]$ . Since the  $g(x)$  is continuous in this interval, we can apply the Intermediate Value Theorem. We see that

$$\begin{aligned} g(-\pi) &= -\frac{5}{2} - 2\pi < 0 \\ g(0) &= \frac{3}{2} \end{aligned}$$

Since  $g(x)$  changes sign in the interval  $[-\pi, 0]$ , we conclude that  $g(x) = 0$  has at least one solution in that interval. Hence  $2x + 2 \cos x - 1 = -\frac{1}{2}$  has at least one solution.

5. (16 points) The following parts are unrelated.

- (a) (i) Find the linearization of  $f(x) = (1+x)^k$  at  $x = 0$ . Note that  $k$  is a constant.  
(ii) Use the linear approximation from the previous part to approximate  $\sqrt[3]{0.90}$ .  
(b) Evaluate  $dy$  if  $y = x^3 - 2x^2 + 1$  when  $x = 2$  and  $dx = 0.2$ .

**Solution:**

- (a) (i) To find the linearization of  $f(x) = (1+x)^k$  at  $x = 0$ . We first find  $f'(x)$ .

$$f'(x) = k(1+x)^{k-1}$$

Next, we evaluate  $f(0) = 1^k = 1$  and  $f'(0) = k \cdot (1)^{k-1} = k$ . Using the formula for a linear approximation,  $L(x) = f(0) + f'(0)(x - 0)$ , we have:

$$L(x) = 1 + k(x - 0) = 1 + kx$$

Therefore, we have that the linearization of  $f(x)$  is:  $L(x) = 1 + kx$

- (ii) To approximate  $\sqrt[3]{0.90}$  using the linear approximation from above, we must have that  $k = \frac{1}{3}$  and

$$1 + x = 0.90 \implies x = -0.1$$

Therefore,  $f(-0.1) \approx L(-0.1) = 1 + \frac{1}{3} \cdot (-0.1) = 1 - \frac{1}{30} = \frac{29}{30}$ .

$$\sqrt[3]{0.90} \approx \frac{29}{30}$$

- (b) Evaluate  $dy$  if  $y = x^3 - 2x^2 + 1$  when  $x = 2$  and  $dx = 0.2$ .

Using differentials we have:

$$dy = (3x^2 - 4x) \cdot (dx)$$

When  $x = 2$  and  $dx = 0.2$ , we have that:

$$\begin{aligned} dy &= (3(2^2) - 4 \cdot 2) \cdot (0.2) \\ &= 4 \cdot 0.2 \\ &= 0.8 \end{aligned}$$

Therefore,  $dy = 0.8$ .