1. (16 pts) Suppose Newton's Method is used to estimate the value of a root of $y = f(x) = \sin x + \cos x + 0.5$.

   (a) Write the general expression for Newton's Method for the specified function $f(x)$. Your answer should be an expression for x_{n+1} in terms of x_n.

   (b) Determine the value of x_1 produced by Newton's Method for an initial value of $x_0 = \pi/2$.

   (c) Find a value of x_0 on the interval $[0, \pi]$ for which Newton's Method would not converge to a solution. Explain why the method would not converge for that value of x_0.

Solution:

(a) The general Newton's Method expression is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, where $f'(x) = \cos x - \sin x$. Therefore,

$$\boxed{x_{n+1} = x_n - \frac{\sin(x_n) + \cos(x_n) + 0.5}{\cos(x_n) - \sin(x_n)}}$$

(b) The result from part (a) can be applied with $n = 1$ and $x_0 = \pi/2$ as follows:

$$\begin{aligned} x_1 &= x_0 - \frac{\sin(x_0) + \cos(x_0) + 0.5}{\cos(x_0) - \sin(x_0)} \\ &= \pi/2 - \frac{\sin(\pi/2) + \cos(\pi/2) + 0.5}{\cos(\pi/2) - \sin(\pi/2)} \\ &= \pi/2 - \frac{1 + 0 + 0.5}{0 - 1} \\ &= \pi/2 + 1.5 = \boxed{\frac{\pi + 3}{2}} \end{aligned}$$

(c) Newton's Method will not converge if $f'(x_0) = 0$.

$$\begin{aligned} f'(x_0) = \cos(x_0) - \sin(x_0) &= 0 \\ \sin(x_0) &= \cos(x_0) \\ \tan(x_0) &= 1 \\ x_0 &= \boxed{\pi/4} \end{aligned}$$

2. (25 pts) Parts (a) and (b) are unrelated.

(a) Evaluate $\int x(2\sqrt{x} - 3)\, dx$.

Solution:

$$\begin{aligned}\int x(2\sqrt{x} - 3)\, dx &= \int (2x^{3/2} - 3x)\, dx \\ &= 2 \cdot \frac{2}{5}x^{5/2} - 3 \cdot \frac{1}{2}x^2 + C \\ &= \boxed{\frac{4}{5}x^{5/2} - \frac{3}{2}x^2 + C}\end{aligned}$$

(b) Suppose the acceleration function of a particle is given by $a(t) = 2\sin t - t + 4$ meters per second squared, and the particle's initial velocity and position are $v(0) = 3$ meters per second and $s(0) = 7$ meters, respectively. Find the particle's position function $s(t)$. Include the correct unit of measurement.

Solution:

$$v(t) = \int a(t)\, dt = -2\cos t - \frac{t^2}{2} + 4t + C_1$$

$$v(0) = 3 = -2\cos(0) - \frac{0^2}{2} + 4 \cdot 0 + C_1$$

$$3 = -2 + C_1 \quad \Rightarrow \quad C_1 = 5$$

$$v(t) = -2\cos t - \frac{t^2}{2} + 4t + 5$$

$$s(t) = \int v(t)\, dt = -2\sin t - \frac{t^3}{6} + 2t^2 + 5t + C_2$$

$$s(0) = 7 = -2\sin(0) - \frac{0^3}{6} + 2 \cdot 0^2 + 5 \cdot 0 + C_2$$

$$7 = 0 + C_2 \quad \Rightarrow \quad C_2 = 7$$

$$s(t) = \boxed{-2\sin t - \frac{t^3}{6} + 2t^2 + 5t + 7 \text{ meters}}$$

3. (24 pts) Parts (a) and (b) are unrelated.

(a) Consider the function $g(x) = x - \dfrac{1}{x}$ on the interval $[1, 5]$.

i. Determine the numerical value of the Riemann sum R_2 for $g(x)$ on $[1, 5]$ using **right** endpoints and 2 equally-sized subintervals. Express your answer as a simplified single fraction.

ii. Write an expression for the general Riemann sum R_n for $g(x)$ on $[1, 5]$ using **right** endpoints and n equally-sized subintervals. Your answer should involve sigma notation and it should be in terms of i (or some other suitable index of summation) and n. Do **not** simplify the expression inside the summation.

Solution:

i. $\Delta x = \dfrac{b-a}{n} = \dfrac{5-1}{2} = 2$

Since right endpoints are being used and the subintervals are of equal size, we have $x_1 = 3$ and $x_2 = 5$.

$$\begin{aligned}
R_2 &= [g(x_1) + g(x_2)]\,\Delta x \\
&= \left(3 - \frac{1}{3} + 5 - \frac{1}{5}\right) \cdot 2 = \left(\frac{9-1}{3} + \frac{25-1}{5}\right) \cdot 2 \\
&= \left(\frac{8}{3} \cdot \frac{5}{5} + \frac{24}{5} \cdot \frac{3}{3}\right) \cdot 2 = \frac{(40+72)(2)}{15} = \boxed{\frac{224}{15}}
\end{aligned}$$

ii. $\Delta x = \dfrac{b-a}{n} = \dfrac{5-1}{n} = \dfrac{4}{n}$

Since right endpoints are being used and the subintervals are of equal size, we have

$$x_i = a + i \cdot \Delta x = 1 + \frac{4i}{n}$$

$$\begin{aligned}
R_n &= \sum_{i=1}^{n} g(x_i) \cdot \Delta x \\
&= \boxed{\sum_{i=1}^{n} \left(1 + \frac{4i}{n} - \frac{1}{1 + \frac{4i}{n}}\right) \cdot \frac{4}{n}}
\end{aligned}$$

(b) Evaluate the following limit using steps (i) and (ii) below: $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^3\cdot\frac{2}{n}$

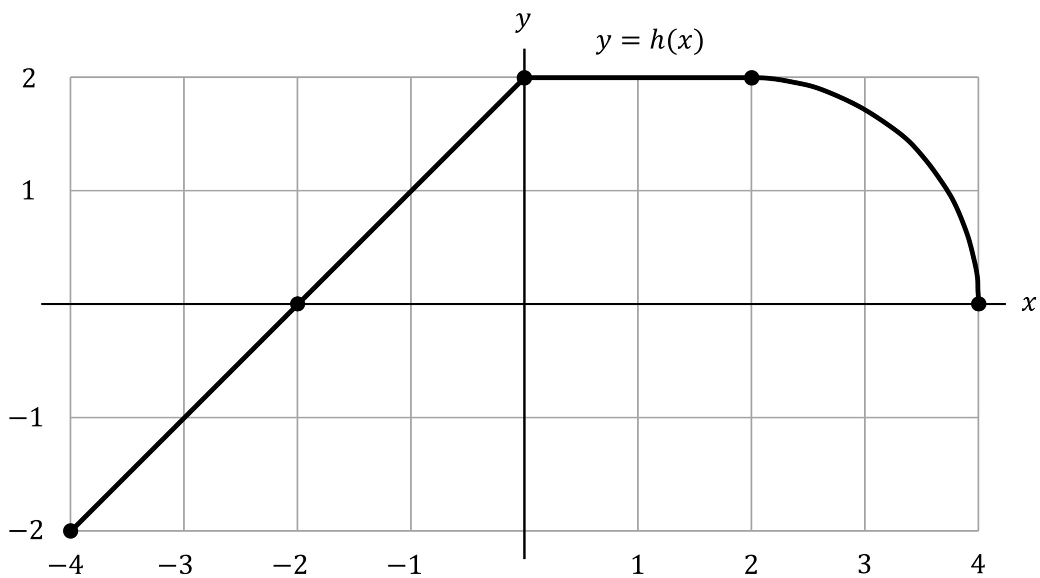
i. Evaluate the given sum to obtain an expression in terms of n that does not involve sigma notation.

ii. Evaluate the limit of the expression obtained in step (i). Do **not** use a dominance of powers argument or L'Hôpital's Rule as part of your limit evaluation.

Solution:

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^3\cdot\frac{2}{n} &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2^3}{n^3}\cdot i^3\cdot\frac{2}{n}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2^4}{n^4}\cdot i^3\\
&= \lim_{n\to\infty}\frac{16}{n^4}\cdot\sum_{i=1}^{n} i^3\\
&= \lim_{n\to\infty}\frac{16}{n^4}\cdot\left(\frac{n(n+1)}{2}\right)^2 \qquad \text{(used formula sheet)}\\
&= \lim_{n\to\infty}\frac{16n^2(n+1)^2}{4n^4}\\
&= \lim_{n\to\infty}\frac{4(n^2+2n+1)}{n^2}\\
&= \lim_{n\to\infty}\left(4+\frac{8}{n}+\frac{4}{n^2}\right)\\
&= 4+0+0=\boxed{4}
\end{aligned}$$

4. (19 pts) The following function graph $y = h(x)$, which consists of two line segments and a quarter circle, is depicted below on the interval $[-4, 4]$.



Evaluate the following integrals.

(a) $\int_{-2}^{2} h(x)\,dx$

Solution:

$$\int_{-2}^{2} h(x)\,dx = \int_{-2}^{0} h(x)\,dx + \int_{0}^{2} h(x)\,dx$$

$=$ area of triangle with vertices $(-2,0), (0,0),$ and $(0,2)$

$+$ area of rectangle with vertices $(0,0), (2,0), (0,2)$ and $(2,2)$

$$= \frac{1}{2} \cdot 2 \cdot 2 + 2 \cdot 2 = \boxed{6}$$

(b) $\int_{-4}^{-1} h(x)\,dx$

Solution:

$$\int_{-4}^{-1} h(x)\,dx = \int_{-4}^{-2} h(x)\,dx + \int_{-2}^{-1} h(x)\,dx$$

$$= \quad - \text{ area of triangle with vertices } (-4,-2), (-4,0), \text{ and } (-2,0)$$

$$= \quad + \text{ area of triangle with vertices } (-2,0), (-1,0), \text{ and } (-1,1)$$

$$= -\frac{1}{2}\cdot 2\cdot 2 + \frac{1}{2}\cdot 1\cdot 1 = \boxed{-\frac{3}{2}}$$

(c) $\int_{-4}^{4} h(x)\,dx$

Solution:

$$\int_{-4}^{4} h(x)\,dx = \int_{-4}^{-2} h(x)\,dx + \int_{-2}^{0} h(x)\,dx + \int_{0}^{2} h(x)\,dx + \int_{2}^{4} h(x)\,dx$$

$$= \quad \text{area of triangle with vertices } (-4,-2), (-4,0), \text{ and } (-2,0)$$

$$+ \text{ area of triangle with vertices } (-2,0), (0,0), \text{ and } (0,2)$$

$$+ \text{ area of rectangle with vertices } (0,0), (2,0), (0,2) \text{ and } (2,2)$$

$$+ \text{ area of quarter circle of radius 2 centered at } (2,0)$$

$$= -\frac{1}{2}\cdot 2\cdot 2 + \frac{1}{2}\cdot 2\cdot 2 + 2\cdot 2 + \frac{1}{4}\cdot\pi\cdot 2^2 = \boxed{4+\pi}$$

5. (16 pts) Evaluate the following derivatives. Do **not** simplify your answers.

(a)

$$\frac{d}{dx}\int_{2x}^{8}\frac{\cos t}{t^2}\,dt$$

Solution:

$$\begin{aligned}
\frac{d}{dx}\int_{2x}^{8}\frac{\cos t}{t^2}\,dt &= -\frac{d}{dx}\int_{8}^{2x}\frac{\cos t}{t^2}\,dt \\
&= -\frac{\cos(2x)}{(2x)^2}\cdot\frac{d}{dx}[2x] \qquad \text{(Fundamental Theorem of Calculus, part 1)} \\
&= -\frac{\cos(2x)}{4x^2}\cdot 2 \\
&= \boxed{-\frac{\cos(2x)}{2x^2}}
\end{aligned}$$

(b)

$$\frac{d}{dx}\int_{x^2}^{\sqrt{x}}\csc^3 t\,dt$$

Solution:

$$\begin{aligned}
\frac{d}{dx}\int_{x^2}^{\sqrt{x}}\csc^3 t\,dt &= \frac{d}{dx}\int_{x^2}^{a}\csc^3 t\,dt + \frac{d}{dx}\int_{a}^{\sqrt{x}}\csc^3 t\,dt \qquad \text{(}a\text{ is a suitable constant)} \\
&= \frac{d}{dx}\int_{a}^{\sqrt{x}}\csc^3 t\,dt - \frac{d}{dx}\int_{a}^{x^2}\csc^3 t\,dt \\
&= \csc^3\left(\sqrt{x}\right)\cdot\frac{d}{dx}[\sqrt{x}] - \csc^3\left(x^2\right)\cdot\frac{d}{dx}[x^2] \qquad \text{(FTC, part 1)} \\
&= \csc^3\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}} - \csc^3\left(x^2\right)\cdot 2x \\
&= \boxed{\frac{1}{2\sqrt{x}}\cdot\csc^3\left(\sqrt{x}\right) - 2x\cdot\csc^3\left(x^2\right)}
\end{aligned}$$